

Gluing Riemannian manifolds with curvature operators at least κ

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Abstract

Let (M_0, g_0) and (M_1, g_1) be smooth Riemannian manifolds with smooth compact boundaries and Riemannian curvature operators $\geq \kappa$ (which means that all eigenvalues of the curvature operators are at least κ), and let M be the Riemannian manifold resulting from gluing M_0 and M_1 along some isometry of their boundaries. The metrics g_0 and g_1 induce a continuous metric g on M . If the sum of the second fundamental forms of the common boundary of M_0 and M_1 with respect to the inward normals is positive semidefinite, then g can be approximated by smooth metrics which have curvature operators almost $\geq \kappa$. An analogous result holds for manifolds with lower bounds on Ricci curvature, scalar curvature (in this case it suffices to assume that the sum of the mean curvatures of the boundary is nonnegative), bi-curvature, isotropic curvature, and flag curvature, respectively.

1 Introduction and statement of results

Gluing Alexandrov spaces of bounded curvature has been studied in a number of works, in particular by Reshetnyak [5] (curvature bounded from above) and Petrunin [4] (curvature bounded from below). The case where the spaces being glued are smooth Riemannian manifolds of sectional curvature at least κ was studied by Kosovskii [2], where he shows that the resulting Alexandrov space has curvature at least κ if and only if the sum of the second fundamental forms of the initial manifolds on their common boundary is positive semidefinite. In this paper we shall examine a similar setup for smooth Riemannian manifolds with smooth compact boundaries and curvature operators $\geq \kappa$. The method being used in [2] can be applied with some modifications.

Let (M, g) be a smooth Riemannian manifold with a smooth metric g , and let $\Lambda^2(TM) \subset TM \otimes TM$ be the bundle of two-vectors over M . Given a point $p \in M$ and a basis $\{e_1, \dots, e_n\}$ of $T_p M$, $\Lambda^2(T_p M)$ is generated by

$$\{e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i < j \leq n\}$$

The metric g induces an inner product \mathcal{I}^g on $\Lambda^2(TM)$, defined by

$$\mathcal{I}^g(e_i \wedge e_j, e_k \wedge e_l) := g_{ik}g_{jl} - g_{jk}g_{il} \tag{1.1}$$

where $g_{ik} = g(e_i, e_k)$. Note that if the vectors e_i are orthonormal with respect to g then the two-vectors $e_i \wedge e_j$ are orthonormal with respect to \mathcal{I}^g . Let $R^g = (R_{ijkl}^g)$ be the Riemannian curvature tensor of g and $R_{ijkl}^g = R^g(e_i, e_j, e_k, e_l)$ (We choose the sign of R_{ijkl}^g such that the sectional curvature of a two-plane spanned by some orthonormal e_i, e_j is given by R_{ijij}^g .) R^g induces a symmetric bilinear form \mathcal{R}^g on $\Lambda^2(TM)$ via

$$\mathcal{R}^g(e_i \wedge e_j, e_k \wedge e_l) = R_{ijkl}^g$$

The Riemannian curvature operator on $\Lambda^2(TM)$, which we shall also denote by \mathcal{R}^g , is defined by the property

$$\mathcal{I}^g(\cdot, \mathcal{R}^g \cdot) = \mathcal{R}^g(\cdot, \cdot)$$

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By $\mathcal{R}^g \geq \kappa \in \mathbb{R}$ (or $\mathcal{R}^g \geq \kappa \mathcal{I}^g$) we mean that all eigenvalues of \mathcal{R}^g are at least κ , or equivalently that

$$\mathcal{R}^g(\alpha, \alpha) \geq \kappa \mathcal{I}^g(\alpha, \alpha)$$

holds for all $\alpha \in \Lambda^2(TM)$.

Let us glue two Riemannian manifolds (M_0, g_0) and (M_1, g_1) along some isometry ϕ of their boundaries (which means we identify $p \in \partial M_0$ and $\phi(p) \in \partial M_1$), and define a metric g on the resulting manifold M by $g|_{M_i} = g_i$, $i = 0, 1$. Due to the isometry of the boundaries, g is continuous, but fails to be C^2 -smooth in general. In this case we can not speak of the Riemannian curvature operator of g in the classical sense. In [2] Kosovskii made use of the fact that nevertheless M can be equipped with a length structure induced by g and instead of bounded sectional curvature in the classical sense one has the notion of bounded curvature in the sense of Alexandrov (see, e.g., [1]). However, there is no analogue of this notion for bounds on the Riemannian curvature operator. We introduce the following definition:

Definition 1.1. *Let M be a Riemannian manifold, equipped with a continuous metric g . We say that the Riemannian curvature operator of g is at least κ iff there exists a family of C^∞ -metrics $(g_{(\delta)})$ on M which converge to g uniformly on every compact subset as δ tends to zero and*

$$\mathcal{R}(g_{(\delta)}) \geq (\kappa - \varepsilon(\delta)) \mathcal{I}(g_{(\delta)})$$

holds with $\varepsilon(\delta) \rightarrow 0$.

In view of the above definitions the main result of this paper is the following

Theorem 1.2. *Let M_0 and M_1 be smooth Riemannian manifolds with (at least C^2 -)smooth metrics g_0 and g_1 and smooth compact boundaries Γ_0 and Γ_1 , respectively. Let L_i be the second fundamental form of Γ_i with respect to the inward normal N_i , $i = 0, 1$. Suppose that there exists an isometry $\phi : \Gamma_0 \rightarrow \Gamma_1$, and let $M = M_0 \cup_\phi M_1$ denote the manifold obtained from gluing M_0 and M_1 along ϕ . Then $\Gamma := \Gamma_0 =_\phi \Gamma_1$ can be seen as a hypersurface of M and we may define $L = L_0 + L_1$ on Γ .*

Let g be the continuous metric on M induced by g_0 and g_1 . Suppose that $\mathcal{R}(g_0)$ and $\mathcal{R}(g_1)$ are at least κ . If L is positive semidefinite, then $\mathcal{R}(g) \geq \kappa$ in the sense of Definition 1.1.

Analogous results hold for manifolds with lower bounds on Ricci curvature, scalar curvature (in this case it suffices to require only that $\text{tr}_g L \geq 0$ on Γ), bi-curvature (the sum of the two smallest eigenvalues of the curvature operator), isotropic curvature and flag curvature, respectively.

Plan of the proof of Theorem 1.2:

We proceed similarly to [2]:

- In Section 2 we sum up auxiliary constructions. We introduce a smooth structure on M relative to which M_0 , M_1 and their common boundary Γ are smooth submanifolds. The metric g on M induced by g_0 and g_1 is continuous.

By modifying the metric g_0 near Γ we construct a new metric g_δ on M_0 . We then define a metric $g_{(\delta)}$ on M by $g_{(\delta)}|_{M_0} = g_\delta$ and $g_{(\delta)}|_{M_1} = g_1$. The coefficients of this metric belong to the Sobolev class $W_{loc}^{2,\infty}$.

All of the constructions in this section have been adopted as it stands from [2], therefore we suppress the proofs to the greatest extent. A more detailed discussion can be found in [2], §§ 3-6.

- In Section 3 we compare the Riemannian curvature operators with respect to g_δ and g_0 on M_0 . This section corresponds with § 7 in [2].
- In Sections 4 and 5 we estimate the curvature operator of g_δ , showing that $\mathcal{R}(g_\delta) \geq \kappa - \varepsilon(\delta)$ holds on M_0 , which implies $\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta)$ a.e. on M for the weakly defined curvature operator of the $W_{loc}^{2,\infty}$ -metric $g_{(\delta)}$.
- In Section 6 we mollify $g_{(\delta)}$ and construct a family of smooth metrics as required in Definition 1.1.

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2 Definitions and auxiliary identities

In [2] Kosovskiĭ introduces a smooth structure on M (Fermi coordinates) relative to which M_0 and M_1 are smooth submanifolds. Moreover, the coefficients of the metric g on M , which is defined by $g|_{M_i} = g_i$, $i = 0, 1$, are continuous (cf. [2], Lemma 3.1). Throughout this work, we will constantly make use of the properties of that structure. To that end, we shall repeat the construction here:

First we cover Γ with coordinate charts (x^1, \dots, x^{n-1}) . If the distance d is small enough, the hypersurfaces $\Gamma(d)$ equidistant to Γ are smooth. For a point $p \in M_0$ near Γ we put $x^n(p) = d(p, \Gamma)$, and $(x^1(p), \dots, x^{n-1}(p))$ are the same as the coordinates of the point of Γ closest to p . On M_1 we repeat this construction with $x^n(p) = -d(p, \Gamma)$. In these coordinates the metric tensor g on M defined above is of the form

$$\begin{pmatrix} g_{1,1} & \cdots & g_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{n-1,1} & \cdots & g_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad (2.1)$$

The coordinate charts (x^1, \dots, x^n) give us a smooth structure on $M = M_0 \cup_\phi M_1$ which we will work with in what follows. All computations near Γ will be carried out in these coordinates, unless noted differently.

Notation 2.1. On M_0 we put $\partial_i = \frac{\partial}{\partial x^i}$ for $1 \leq i \leq n-1$ and $N = \frac{\partial}{\partial x^n}$, where (x^1, \dots, x^n) is the coordinate chart introduced above. Note that $N(p)$ is a well defined smooth vectorfield near Γ , which is normal to the hypersurface $\Gamma(d(p))$ equidistant to Γ and containing p .

The following lemma uses the above construction to smoothly extend the metric g_1 from M_1 to a small neighborhood of Γ on M_0 .

Lemma 2.2 ([2], Lemma 3.1). *The metric g_1 smoothly extends to a small neighborhood of Γ in M_0 in such a way that the hypersurfaces equidistant to Γ with respect to the extended metric g'_1 and the metric g_0 coincide.*

Proof. In coordinates defined above the metric g_1 on M_1 is of the same form as in (2.1). Locally in a small enough coordinate neighborhood U of some point of Γ we may smoothly extend $(g_1)_{ij}$, $1 \leq i, j \leq n-1$ to $U \cap M_0$ and put $(g'_1)_{in} = \delta_{in}$. We then cover Γ by finitely many such neighborhoods and define g'_1 near Γ using a subordinate partition of unity. One easily checks that the obtained metric has the desired property. \square

Throughout this work we will use the following

Notation 2.3. Given a $(0, 2)$ tensor A on TM we denote by \mathbf{A} the corresponding linear endomorphism of TM satisfying

$$A(v, w) = \langle v, \mathbf{A}w \rangle_g$$

If $\{e_1, \dots, e_n\}$ is a basis of TM at some point $p \in M$ and $\mathbf{A}e_i = \mathbf{A}_i^j e_j$, then $\mathbf{A}_i^j = A_{ki} g^{kl}$, where $A_{ki} = A(e_k, e_i)$ and $(g^{kl})_{1 \leq k, l \leq n}$ is the inverse of the matrix $(g(e_k, e_l))_{1 \leq k, l \leq n}$. The operator \mathbf{A} is self-adjoint iff the tensor A is symmetric.

Definition, Lemma 2.4 (The operator \mathbf{L} , cf. [2], 3.4 and 3.5). *Let L be the sum of the second fundamental forms on Γ with respect to the inward normals on M_0 and M_1 (or the difference of the second fundamental forms with respect to the common normal N), and let \mathbf{L} be the corresponding selfadjoint operator on $T\Gamma$, i.e. $L(\cdot, \cdot) = \langle \cdot, \mathbf{L}\cdot \rangle_0$.*

In a small neighborhood of Γ the operator \mathbf{L} extends to TM_0 so that $\mathbf{L}N = 0$ and $\nabla_N \mathbf{L} = 0$.

Proof. For a point $p \in \Gamma$ we may extend \mathbf{L} to $T_p M_0$ by linearity such that $\mathbf{L}N = 0$, and for $X \in TM_0$ we use parallel transportation P along the integral curves of the vector field N and put $\mathbf{L}X := P^{-1} \mathbf{L}PX$ \square

Note that if the initial operator is positive semidefinite, then so is its extension:

$$\langle X, \mathbf{L}X \rangle_0 = \langle X, P^{-1} \mathbf{L}PX \rangle_0 = \langle PX, \mathbf{L}PX \rangle_0 \geq 0$$

The following $C^\infty([0, \infty), \mathbb{R})$ -functions will be used to modify the metric g_0 near Γ :

Definition 2.5 (Auxiliary functions f_δ , F_δ and \mathcal{F}_δ , cf. [2], 3.3). *Let f_δ be a function on $\mathbb{R}_{\geq 0}$ with the following properties:*

$$\begin{aligned} f_\delta(x) &= 1 - \frac{x}{\delta^4} & \text{if } x \in [0, \delta^4] \\ -\delta^2 \leq f_\delta \leq 0 \text{ and } f'_\delta(x) &\leq \delta & \text{if } x \in [\delta^4, \delta] \\ f_\delta(x) &= 0 & \text{if } x \in [\delta, +\infty) \end{aligned}$$

and

$$\int_0^\delta f_\delta(t) dt = 0$$

We put

$$F_\delta(x) := \int_0^x f_\delta(t) dt$$

$$\mathcal{F}_\delta(x) := \int_0^x F_\delta(t) dt$$

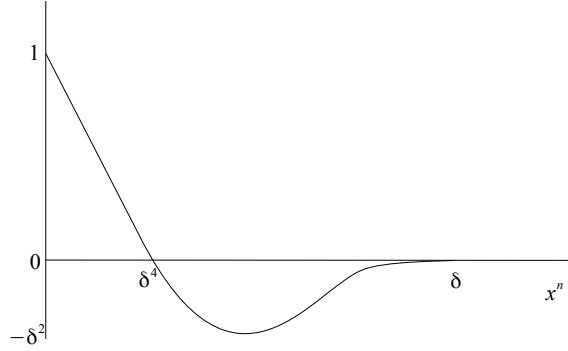


Figure 1: The function f_δ

Notation 2.6 (Projection operators). *Let*

$$\mathbf{P}^T : TM_0 \rightarrow T\Gamma(d) \subset TM_0$$

and

$$\mathbf{P}^N : TM_0 \rightarrow (T\Gamma(d))^\perp \subset TM_0$$

be the projection operators. The coefficients of the corresponding $(0, 2)$ -tensors (with respect to the coordinates chosen above) are

$$(P^T)_{ij} = \begin{pmatrix} (g_{ij})_{1 \leq i, j \leq n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

and $(P^N)_{ij} = \delta_{in} \delta_{jn}$

Definition 2.7 (The modified metric g_δ , [2], 3.6). Let \mathbf{I} denote the identity on TM_0 . We define the self-adjoint endomorphism \mathbf{G}_δ by

$$\mathbf{G}_\delta = \mathbf{I} + 2F_\delta(x^n)\mathbf{L} - 2C\mathcal{F}_\delta(x^n)\mathbf{P}^T \quad (2.2)$$

and the modified inner product $\langle \cdot, \cdot \rangle_\delta$ on TM_0 by

$$\langle \cdot, \cdot \rangle_\delta = \langle \cdot, \mathbf{G}_\delta \cdot \rangle_0$$

i.e. in coordinates we have

$$g_{ij}^\delta = g_{ij}^0 + 2F_\delta(x^n)L_{ij} - 2C\mathcal{F}_\delta(x^n)(P^T)_{ij}$$

The constant C the definition of \mathbf{G}_δ is to be chosen later. Note that the operator \mathbf{G}_δ is well defined globally on M_0 , since by our choice of coordinates we have $x^n(p) = d_{g_0}(p, \Gamma)$ for a point p near Γ , and Γ is compact by assumption.

Remark 2.8. \mathbf{G}_δ has the following properties:

(i) As δ tends to zero, \mathbf{G}_δ converges to \mathbf{I} uniformly on M_0 .

(ii) The coefficients of the metric $g_{(\delta)}$ which is defined as g_δ on M_0 and g_1 on M_1 belong to $W_{loc}^{2,\infty}$

(iii) The hypersurfaces $\Gamma(d_{g_0})$ and $\Gamma(d_{g_\delta})$ coincide.

Proof. (i): \mathbf{L} and \mathbf{P}^T are bounded near Γ , and $F_\delta, \mathcal{F}_\delta \rightarrow 0$ uniformly as $\delta \rightarrow 0$.

(ii): On Γ we have

$$\partial_k g_{ij}^\delta = \partial_k g_{ij}^0 = \partial_k g_{ij}^1$$

for $k = 1, \dots, n-1$, since $g_0 = g_1$ on Γ . In a point $p \in \Gamma$, using $L_{ij}^0 = -\langle \nabla_{\partial_i}^0 N, \partial_j \rangle_0$ and $L_{ij}^1 = \langle \nabla_{\partial_i}^1 N, \partial_j \rangle_1$ one computes

$$\partial_n g_{ij}^0 = -2L_{ij}^0$$

and

$$\partial_n g_{ij}^1 = 2L_{ij}^1$$

Thus, on Γ we have

$$\partial_n g_{ij}^\delta = \partial_n g_{ij}^0 + 2L_{ij} = 2(L_{ij} - L_{ij}^0) = 2L_{ij}^1 = \partial_n g_{ij}^1$$

which implies that the first derivatives of $g_{(\delta)}$ are continuous on M . Since $\Gamma \subset M$ is a smooth submanifold and $g_{(\delta)}$ is smooth on M_0 and M_1 , respectively, we have $g_{(\delta)} \in W_{loc}^{2,\infty}$

(iii) Note that $g_\delta(\partial_i, N) = \delta_{in} = g_0(\partial_i, N)$ due to $\mathbf{L}N = 0 = \mathbf{P}^T N$. Therefore N is also normal with respect to g_δ to the hypersurfaces $\Gamma(d_{g_\delta})$, which implies $d_{g_\delta}(\cdot, \Gamma) = d_g(\cdot, \Gamma)$. \square

Definition 2.9. For two endomorphisms $\mathbf{S}_\delta, \mathbf{T}_\delta$ of TM_0 which depend on δ we say that

$$\mathbf{S}_\delta \approx \mathbf{T}_\delta$$

iff $\mathbf{S}_\delta|_\Gamma = \mathbf{T}_\delta|_\Gamma$ and all eigenvalues of $\mathbf{S}_\delta - \mathbf{T}_\delta$ uniformly tend to zero on compact subsets of M_0 as $\delta \rightarrow 0$. For two vectorfields X_δ, Y_δ on M_0 we say that $X_\delta \approx Y_\delta$ iff $X_\delta|_\Gamma = Y_\delta|_\Gamma$ and $\|X_\delta - Y_\delta\|_0 \rightarrow 0$ uniformly on compact subsets as $\delta \rightarrow 0$.

Note that $\mathbf{S}_\delta \approx \mathbf{T}_\delta$ ($X_\delta \approx Y_\delta$) holds iff in local coordinates $(S_\delta)_{ij} = (T_\delta)_{ij}$ on Γ and $|(S_\delta)_{ij} - (T_\delta)_{ij}| \rightarrow 0$ ($X_\delta^i = Y_\delta^i$ on Γ and $|X_\delta^i - Y_\delta^i| \rightarrow 0$).

Lemma 2.10 (Auxiliary identities, cf. [2], Lemma 6.1, 6.2, 6.3). Let

$$X, Y \in \{\partial_1, \dots, \partial_{n-1}\} \subset T\Gamma(d) \subset TM_0$$

and

$$N = \partial_n \in (T\Gamma(d))^\perp \subset TM_0$$

Then the following (approximate and exact) identities hold

$$\begin{aligned} \mathbf{G}_\delta &\approx \mathbf{I}, \quad \nabla_X \mathbf{G}_\delta \approx 0, \quad \nabla_N \mathbf{G}_\delta \approx 2f_\delta(x^n)\mathbf{L} \\ \nabla_X \nabla_N \mathbf{G}_\delta &\approx 2f_\delta(x^n)\nabla_X \mathbf{L} \\ \nabla_N \nabla_N \mathbf{G}_\delta &\approx 2f'_\delta(x^n)\mathbf{L} - 2Cf_\delta(x^n)\mathbf{P}^T \end{aligned} \quad (2.3)$$

$$\langle \nabla_X^\delta N, Y \rangle_\delta = \langle \nabla_N^\delta X, Y \rangle_\delta = \frac{1}{2} (\langle \nabla_N X, \mathbf{G}_\delta Y \rangle + \langle X, \mathbf{G}_\delta \nabla_N Y \rangle + \langle X, (\nabla_N \mathbf{G}_\delta) Y \rangle) \quad (2.4)$$

$$\nabla_N^\delta N = 0 \quad (2.5)$$

$$\nabla_N^\delta X = \nabla_X^\delta N \approx \nabla_X N + f_\delta(x^n)\mathbf{L}X \quad (2.6)$$

$$\mathbf{P}^T(\nabla_X^\delta Y) \approx \mathbf{P}^T(\nabla_X Y) \quad (2.7)$$

Proof. Detailed proofs of these identities are given in [2]. For the convenience of the reader we prove one of the identities in (2.3). $\nabla_X \mathbf{G}_\delta \approx 0$:

By definition of \mathbf{G}_δ we have

$$\nabla_X \mathbf{G}_\delta = \nabla_X \mathbf{I} + 2\nabla_X(F_\delta(x^n)\mathbf{L}) - 2C\nabla_X(\mathcal{F}_\delta(x^n)\mathbf{P}^T) \quad (2.8)$$

Let $\xi \in TM_0$. We compute

$$\nabla_X \mathbf{I}(\xi) = \nabla_X(\mathbf{I}\xi) - \mathbf{I}(\nabla_X \xi) = 0$$

and

$$\begin{aligned} \nabla_X(F_\delta(x^n)\mathbf{L})\xi &= \nabla_X(F_\delta(x^n)\mathbf{L}\xi) - F_\delta(x^n)\mathbf{L}\nabla_X \xi \\ &= X(F_\delta(x^n))\mathbf{L}\xi + F_\delta(x^n)\nabla_X(\mathbf{L}\xi) - F_\delta(x^n)\mathbf{L}\nabla_X \xi \end{aligned}$$

The first term of the last expression vanishes since $X \in \{\partial_1, \dots, \partial_{n-1}\}$ and F_δ only depends on x^n . The next two terms tend to zero as $\delta \rightarrow 0$ by definition of F_δ . By a similar computation one verifies that the last term in (2.8) is ≈ 0 as well.

□

3 The Riemannian curvature operator of g_δ

In this section we compare the Riemannian curvature operators of g_δ and g_0 on M_0 (cf. §§ 7-8 of [2]).

Let us first recall that given a finite dimensional vectorspace V one has the following connection between $(0, 4)$ -tensors on V and linear operators and bilinear forms on $\Lambda^2 V$: Any $(0, 4)$ -tensor $\{T_{ijkl}\}$ which is antisymmetric in i, j and k, l , respectively, induces a bilinear form \mathcal{T} on $\Lambda^2 V$ via

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) := T(e_i, e_j, e_k, e_l) = T_{ijkl}$$

where e_1, \dots, e_n is some basis of V . The antisymmetries of T ensure that

$$\mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) = -\mathcal{T}(e_j \wedge e_i, e_k \wedge e_l) = -\mathcal{T}(e_i \wedge e_j, e_l \wedge e_k)$$

If in addition $T_{ijkl} = T_{klij}$ then the induced bilinearform \mathcal{T} is symmetric. For arbitrary $\alpha, \beta \in \Lambda^2 V$, $\alpha = \sum_{i < j} \alpha^{ij} e_i \wedge e_j = \alpha^{ij} e_i \otimes e_j$, $\beta = \sum_{i < j} \beta^{ij} e_i \wedge e_j = \beta^{ij} e_i \otimes e_j$ ($\alpha^{ij} = -\alpha^{ji}$ and $\beta^{ij} = -\beta^{ji}$) one computes

$$\mathcal{T}(\alpha, \beta) = \frac{1}{4} T_{ijkl} \alpha^{ij} \beta^{kl}$$

(here and in hat follows we make use of the summation convention). As mentioned in the introduction, an inner product g on V induces an inner product \mathcal{I}^g on $\Lambda^2 V$:

$$\mathcal{I}^g(e_i \wedge e_j, e_k \wedge e_l) = g_{ik} g_{jl} - g_{jk} g_{il}$$

Using this inner product we may identify linear operators and bilinear forms on $\Lambda^2 V$ by putting

$$\mathcal{I}^g(e_i \wedge e_j, \mathcal{T}(e_k \wedge e_l)) = \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l)$$

The bilinear form is symmetric iff the operator is self-adjoint.

Conversely, any bilinear form \mathcal{T} on $\Lambda^2 V$ (or the corresponding linear operator) induces a $(0, 4)$ -tensor on V via

$$T(e_i, e_j, e_k, e_l) := \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l) = \mathcal{I}^g(e_i \wedge e_j, \mathcal{T}(e_k \wedge e_l))$$

The such defined tensor has the symmetries $T_{ijkl} = -T_{jikl} = -T_{ijlk}$, and if in addition the bilinear form is symmetric, then we also have $T_{ijkl} = T_{klij}$.

In view of these identifications, in what follows we will often switch between operators and bilinear forms on $\Lambda^2(TM)$ and $(0, 4)$ -tensors on TM .

In Section 7.1 we will use the following

Lemma 3.1. *Let \mathcal{T} be a bilinear form on $\Lambda^2 V$ and (T_{ijkl}) the corresponding $(0, 4)$ -tensor on V . If \mathcal{T} is positive semidefinite, then so is the bilinear form $\text{tr}_{24} T := g^{jl} T(\cdot, e_j, \cdot, e_l) : V \times V \rightarrow \mathbb{R}$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of V such that $g_{ij} = g(e_i, e_j) = \delta_{ij}$. Let $\xi = \xi^k e_k \in V$. For every $1 \leq j \leq n$ we define the 2-vector $\alpha_j = \xi^k e_k \wedge e_j \in \Lambda^2 V$. By assumption $\mathcal{T}(\alpha_j, \alpha_j) \geq 0$. We compute

$$\begin{aligned} \text{tr}_{24} T(\xi, \xi) &= g^{jl} T(\xi, e_j, \xi, e_l) = \sum_j \xi^k \xi^l T(e_k, e_j, e_l, e_j) \\ &= \sum_j \xi^k \xi^l \mathcal{T}(e_k \wedge e_j, e_l \wedge e_j) = \sum_j \mathcal{T}(\alpha_j, \alpha_j) \geq 0 \end{aligned}$$

□

Note that this lemma also holds if we replace $\text{tr}_{24} T$ by $\text{tr}_{13} T$. We will also make use of the Kulkarni-Nomizu product on $\text{End}(TM)$, which is defined as follows:

The Kulkarni-Nomizu product of two linear endomorphisms \mathbf{A}, \mathbf{B} of V is the linear endomorphism $\mathbf{A} \wedge \mathbf{B} : \Lambda^2 V \rightarrow \Lambda^2 V$, which is defined as

$$(\mathbf{A} \wedge \mathbf{B})(e_i \wedge e_j) := \frac{1}{2} (\mathbf{A}(e_i) \wedge \mathbf{B}(e_j) + \mathbf{B}(e_i) \wedge \mathbf{A}(e_j))$$

for basis vectors $e_i \wedge e_j$, and extends to $\Lambda^2 V$ by linearity. The factor $\frac{1}{2}$ ensures that we have $\text{id}_V \wedge \text{id}_V = \text{id}_{\Lambda^2 V}$. The corresponding bilinear form on $\Lambda^2 V$ is given by

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B}(e_i \wedge e_j, e_k \wedge e_l) &:= \mathcal{I}^g(e_i \wedge e_j, (\mathbf{A} \wedge \mathbf{B})(e_k \wedge e_l)) \\ &= \frac{1}{2} (A_{ik} B_{jl} - A_{jk} B_{il} + B_{ik} A_{jl} - B_{jk} A_{il}) \end{aligned}$$

where A, B are the bilinear forms on V corresponding with \mathbf{A}, \mathbf{B} (cf. Notation 2.3). Note that the induced $(0, 4)$ -tensor $\{(\mathbf{A} \wedge \mathbf{B})_{ijkl}\}$ is antisymmetric in i, j and k, l , respectively. If in addition A and B are symmetric, then we also have the symmetry $(\mathbf{A} \wedge \mathbf{B})_{ijkl} = (\mathbf{A} \wedge \mathbf{B})_{klij}$.

Throughout this work we will frequently make use of the following

Lemma 3.2. *Let \mathbf{A}, \mathbf{B} be two self-adjoint endomorphisms of (V, g) . If $\mathbf{A}, \mathbf{B} \geq 0$ (in the sense of eigenvalues) then $\mathbf{A} \wedge \mathbf{B} \geq 0$.*

Proof. It suffices to show that $(\mathbf{A} \wedge \mathbf{B})(\alpha, \alpha) \geq 0$ for any $\alpha \in \Lambda^2 V$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of (V, g) such that $A_{ij} = \lambda_i \delta_{ij}$ with respect to this basis (where $\lambda_i \geq 0$ by assumption), and $\alpha = \sum_{i < j} \alpha^{ij} e_i \wedge e_j = \alpha^{ij} e_i \otimes e_j \in \Lambda^2 V$. As mentioned above, a bilinear form \mathcal{T} induced by a $(0, 4)$ -tensor T satisfies

$$\mathcal{T}(\alpha, \beta) = \frac{1}{4} T_{ijkl} \alpha^{ij} \beta^{kl}$$

where $T_{ijkl} := \mathcal{T}(e_i \wedge e_j, e_k \wedge e_l)$. We compute

$$\begin{aligned} (\mathbf{A} \wedge \mathbf{B})(\alpha, \alpha) &= \frac{1}{8} (A_{ik} B_{jl} \alpha^{ij} \alpha^{kl} - A_{jk} B_{il} \alpha^{ij} \alpha^{kl} + B_{ik} A_{jl} \alpha^{ij} \alpha^{kl} - B_{jk} A_{il} \alpha^{ij} \alpha^{kl}) \\ &= \frac{1}{2} A_{ik} B_{jl} \alpha^{ij} \alpha^{kl} \\ &= \frac{1}{2} \lambda_i B_{jl} \alpha^{ij} \alpha^{il} \geq 0 \end{aligned}$$

where we used $\alpha^{ij} = -\alpha^{ji}$ and the fact that for every fixed i we have $B_{jl} \alpha^{ij} \alpha^{il} \geq 0$ by assumption. \square

Let us now consider the Riemannian curvature operator of g_δ . For ease of notation here and in what follows we shall omit the index 0 for quantities related to M_0 . For example, we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_0$ and \mathcal{R} for \mathcal{R}_0 . We define $\mathcal{S}_\delta \approx \mathcal{T}_\delta$ for selfadjoint operators on $\Lambda^2(TM_0)$ in a similar way as in Definition 2.9. The main result of this section is

Theorem 3.3. *Let $\mathcal{R}_\delta = \mathcal{R}(g_\delta)$. Then*

$$\mathcal{R}_\delta \approx \mathcal{R} - f_\delta^2 \mathcal{A} + f_\delta \mathcal{B} - 2f'_\delta \mathcal{L} + 2f_\delta^2 \mathcal{L}^2 + 2C f_\delta \hat{\mathcal{L}} \quad (3.1)$$

holds on M_0 , where

$$\begin{aligned} \mathcal{A} &:= \mathbf{L} \wedge \mathbf{L} \\ \mathcal{L} &:= \mathbf{L} \wedge \mathbf{P}^N \\ \mathcal{L}^2 &:= \mathbf{L}^2 \wedge \mathbf{P}^N \\ \hat{\mathcal{L}} &:= \mathbf{P}^T \wedge \mathbf{P}^N \end{aligned}$$

(cf. Notation 2.6 for the definitions of \mathbf{P}^T and \mathbf{P}^N), and \mathcal{B} is a smooth operator on $\Lambda^2(TM)$ depending on \mathbf{L} which we will define later.

Lemma 3.4. *For $i, j, k, l \in \{1, \dots, n-1\}$ we have*

$$R_{ijkl}^\delta \approx R_{ijkl} - f_\delta^2 (\mathbf{L} \wedge \mathbf{L})_{ijkl} - 2f_\delta (\mathbf{L} \wedge \nabla N)_{ijkl} \quad (3.2)$$

where ∇N is the endomorphism $X \in TM \mapsto \nabla_X N \in TM$ (recall that N is the unit vector field orthogonal to the hypersurfaces of M_0 equidistant to Γ , cf. Notation 2.1).

Proof. We proceed as in [2], Lemma 7.1. Let $p \in M_0$ be a point near Γ and $d = \text{dist}(x, \Gamma) = x^n(p)$. Let $k, l \leq n-1$. Recall that by Definition 2.2 we have

$$g_{kl}^\delta = g_{kl} + 2F_\delta(x^n) L_{kl} - 2C \mathcal{F}_\delta(x^n) g_{kl}$$

Therefore, for $i, j \leq n-1$

$$\partial_i g_{kl}^\delta \approx \partial_i g_{kl}$$

and

$$\partial_i \partial_j g_{kl}^\delta \approx \partial_i \partial_j g_{kl}$$

and thus

$$R_{\Gamma(d)}^\delta \approx R_{\Gamma(d)}$$

Using the Gauss theorem and (2.6) we compute in p

$$\begin{aligned}
R_{ijkl}^\delta &= \langle R^\delta(\partial_i, \partial_j)\partial_k, \partial_l \rangle_\delta \\
&= \langle R_{\Gamma(d)}^\delta(\partial_i, \partial_j)\partial_k, \partial_l \rangle_\delta - \langle \nabla_{\partial_i}^\delta N, \partial_k \rangle_\delta \langle \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta + \langle \nabla_{\partial_j}^\delta N, \partial_k \rangle_\delta \langle \nabla_{\partial_i}^\delta N, \partial_l \rangle_\delta \\
&\approx \langle R_{\Gamma(d)}^\delta(\partial_i, \partial_j)\partial_k, \partial_l \rangle - (\langle \nabla_{\partial_i} N, \partial_k \rangle + f_\delta \langle \partial_i, \mathbf{L}\partial_k \rangle)(\langle \nabla_{\partial_j} N, \partial_l \rangle + f_\delta \langle \partial_j, \mathbf{L}\partial_l \rangle) \\
&\quad + (\langle \nabla_{\partial_j} N, \partial_k \rangle + f_\delta \langle \partial_j, \mathbf{L}\partial_k \rangle)(\langle \nabla_{\partial_i} N, \partial_l \rangle + f_\delta \langle \partial_i, \mathbf{L}\partial_l \rangle) \\
&= \langle R_{\Gamma(d)}^\delta(\partial_i, \partial_j)\partial_k, \partial_l \rangle - \langle \nabla_{\partial_i} N, \partial_k \rangle \langle \nabla_{\partial_j} N, \partial_l \rangle + \langle \nabla_{\partial_j} N, \partial_k \rangle \langle \nabla_{\partial_i} N, \partial_l \rangle \\
&\quad - f_\delta^2 (\langle \partial_i, \mathbf{L}\partial_k \rangle \langle \partial_j, \mathbf{L}\partial_l \rangle - \langle \partial_j, \mathbf{L}\partial_k \rangle \langle \partial_i, \mathbf{L}\partial_l \rangle) \\
&\quad - f_\delta (\langle \partial_i, \mathbf{L}\partial_k \rangle \langle \nabla_{\partial_j} N, \partial_l \rangle - \langle \partial_j, \mathbf{L}\partial_k \rangle \langle \nabla_{\partial_i} N, \partial_l \rangle + \langle \nabla_{\partial_i} N, \partial_k \rangle \langle \partial_j, \mathbf{L}\partial_l \rangle - \langle \nabla_{\partial_j} N, \partial_k \rangle \langle \partial_i, \mathbf{L}\partial_l \rangle)
\end{aligned}$$

□

Lemma 3.5. For $i, j, l \in \{1, \dots, n-1\}$ we have

$$R_{ijnl}^\delta \approx R_{ijnl} + f_\delta (\langle \partial_i, (\nabla_{\partial_j} \mathbf{L})\partial_l \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L})\partial_l \rangle) \quad (3.3)$$

Proof. We proceed as in [2], Lemma 7.3. Let $i, j, l \in \{1, \dots, n-1\}$. By definition of the Riemannian curvature tensor we have

$$\begin{aligned}
\langle R^\delta(\partial_i, \partial_j)\partial_n, \partial_l \rangle_\delta &= \langle \nabla_{\partial_j}^\delta \nabla_{\partial_i}^\delta N, \partial_l \rangle_\delta - \langle \nabla_{\partial_i}^\delta \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta \\
&= \partial_j \langle \nabla_{\partial_i}^\delta N, \partial_l \rangle_\delta - \partial_i \langle \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta - \langle \nabla_{\partial_i}^\delta N, \nabla_{\partial_j}^\delta \partial_l \rangle_\delta + \langle \nabla_{\partial_j}^\delta N, \nabla_{\partial_i}^\delta \partial_l \rangle_\delta
\end{aligned} \quad (3.4)$$

1) For the first two terms on the right hand side we compute using (2.4):

$$\begin{aligned}
&\partial_j \langle \nabla_{\partial_i}^\delta N, \partial_l \rangle_\delta - \partial_i \langle \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta \\
&= \frac{1}{2} \partial_j (\langle \nabla_N \partial_i, \mathbf{G}_\delta \partial_l \rangle + \langle \partial_i, \mathbf{G}_\delta \nabla_N \partial_l \rangle + \langle \partial_i, (\nabla_N \mathbf{G}_\delta) \partial_l \rangle) \\
&\quad - \frac{1}{2} \partial_i (\langle \nabla_N \partial_j, \mathbf{G}_\delta \partial_l \rangle + \langle \partial_j, \mathbf{G}_\delta \nabla_N \partial_l \rangle + \langle \partial_j, (\nabla_N \mathbf{G}_\delta) \partial_l \rangle)
\end{aligned}$$

After termwise differentiation we get three different types of terms:

a) Terms in which \mathbf{G}_δ is not differentiated: Since $\mathbf{G}_\delta \approx \mathbf{I}$, their sum is approximately equal to

$$\partial_j \langle \nabla_{\partial_i} N, \partial_l \rangle - \partial_i \langle \nabla_{\partial_j} N, \partial_l \rangle$$

b) Terms in which \mathbf{G}_δ is differentiated only with respect to ∂_i are approximately equal to 0 by (2.3) since we have $i \in \{1, \dots, n-1\}$.

c) Terms which involve mixed derivatives of \mathbf{G}_δ with respect to both ∂_i and N . In view of (2.3) their sum is

$$\begin{aligned}
&\approx f_\delta (\langle \nabla_{\partial_j} \partial_i, \mathbf{L}\partial_l \rangle + \langle \partial_i, (\nabla_{\partial_j} \mathbf{L})\partial_l \rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j} \partial_l) \rangle) \\
&\quad - \langle \nabla_{\partial_i} \partial_j, \mathbf{L}\partial_l \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L})\partial_l \rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i} \partial_l) \rangle) \\
&= f_\delta (\langle \partial_i, (\nabla_{\partial_j} \mathbf{L})\partial_l \rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j} \partial_l) \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L})\partial_l \rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i} \partial_l) \rangle)
\end{aligned}$$

where we used that ∂_i and ∂_j commute. Combining a), b) and c) gives us

$$\begin{aligned}
\partial_j \langle \nabla_{\partial_i}^\delta N, \partial_l \rangle_\delta - \partial_i \langle \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta &\approx \partial_j \langle \nabla_{\partial_i} N, \partial_l \rangle - \partial_i \langle \nabla_{\partial_j} N, \partial_l \rangle \\
&\quad + f_\delta (\langle \partial_i, (\nabla_{\partial_j} \mathbf{L})\partial_l \rangle + \langle \partial_i, \mathbf{L}(\nabla_{\partial_j} \partial_l) \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L})\partial_l \rangle - \langle \partial_j, \mathbf{L}(\nabla_{\partial_i} \partial_l) \rangle)
\end{aligned} \quad (3.5)$$

2) Let us now consider the last two terms on the right hand side of (3.4). Since $\langle \nabla_{\partial_i}^\delta N, N \rangle_\delta = 0$ we have

$$\langle \nabla_{\partial_i}^\delta N, \nabla_{\partial_j}^\delta \partial_l \rangle_\delta = \langle \nabla_{\partial_i}^\delta N, \mathbf{P}^T(\nabla_{\partial_j}^\delta \partial_l) \rangle_\delta$$

Therefore, in view of (2.6) and (2.7)

$$\begin{aligned}
& -\langle \nabla_{\partial_i}^\delta N, \nabla_{\partial_j}^\delta \partial_l \rangle_\delta + \langle \nabla_{\partial_j}^\delta N, \nabla_{\partial_i}^\delta \partial_l \rangle_\delta \\
= & -\langle \nabla_{\partial_i}^\delta N, \mathbf{P}^T(\nabla_{\partial_j}^\delta \partial_l) \rangle_\delta + \langle \nabla_{\partial_j}^\delta N, \mathbf{P}^T(\nabla_{\partial_i}^\delta \partial_l) \rangle_\delta \\
\approx & -(\langle \nabla_{\partial_i} N, \mathbf{P}^T(\nabla_{\partial_j} \partial_l) \rangle + f_\delta \langle \mathbf{L} \partial_i, \mathbf{P}^T(\nabla_{\partial_j} \partial_l) \rangle) + (\langle \nabla_{\partial_j} N, \mathbf{P}^T(\nabla_{\partial_i} \partial_l) \rangle + f_\delta \langle \mathbf{L} \partial_j, \mathbf{P}^T(\nabla_{\partial_i} \partial_l) \rangle) \\
\approx & -\langle \nabla_{\partial_i} N, \nabla_{\partial_j} \partial_l \rangle + \langle \nabla_{\partial_j} N, \nabla_{\partial_i} \partial_l \rangle + f_\delta (\langle \mathbf{L} \partial_j, \nabla_{\partial_i} \partial_l \rangle - \langle \mathbf{L} \partial_i, \nabla_{\partial_j} \partial_l \rangle)
\end{aligned} \tag{3.6}$$

where in the last line we used $\langle \mathbf{L} \partial_i, N \rangle = \langle \partial_i, \mathbf{L} N \rangle = 0$.

Combining (3.5) and (3.6) we obtain the desired result. \square

Lemma 3.6. For $j, l \in \{1, \dots, n-1\}$ we have

$$\begin{aligned}
R_{njnl}^\delta \approx & R_{njnl} - 2f'_\delta(\mathbf{L} \wedge \mathbf{P}^N)_{njnl} + 2f_\delta^2(\mathbf{L}^2 \wedge \mathbf{P}^N)_{njnl} + 2Cf_\delta(\mathbf{P}^T \wedge \mathbf{P}^N)_{njnl} \\
& - f_\delta(\langle \mathbf{L} \partial_j, \nabla_{\partial_l} N \rangle + \langle \nabla_{\partial_j} N, \mathbf{L} \partial_l \rangle)
\end{aligned}$$

Proof. We proceed as in [2], Lemma 7.2. Using Lemma 2.10 we compute

$$\begin{aligned}
R_{njnl}^\delta &= \langle R^\delta(N, \partial_j)N, \partial_l \rangle_\delta \\
&= \langle \nabla_{\partial_j}^\delta \underbrace{\nabla_N^\delta N}_{=0}, \partial_l \rangle_\delta - \langle \nabla_N^\delta \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta \\
&= -N \langle \nabla_{\partial_j}^\delta N, \partial_l \rangle_\delta + \langle \nabla_{\partial_j}^\delta N, \nabla_N^\delta \partial_l \rangle_\delta \\
&\stackrel{(2.4), (2.6)}{\approx} -N \left[\frac{1}{2} (\langle \nabla_N \partial_j, \mathbf{G}_\delta \partial_l \rangle + \langle \partial_j, \mathbf{G}_\delta(\nabla_N \partial_l) \rangle + \langle \partial_j, (\nabla_N \mathbf{G}_\delta) \partial_l \rangle) \right] \\
&\quad + \langle \nabla_N \partial_j + f_\delta \mathbf{L} \partial_j, \nabla_N \partial_l + f_\delta \mathbf{L} \partial_l \rangle \\
&\stackrel{(2.3)}{\approx} \langle R(N, \partial_j)N, \partial_l \rangle - f'_\delta \langle \partial_j, \mathbf{L} \partial_l \rangle + f_\delta^2 \langle \mathbf{L} \partial_j, \mathbf{L} \partial_l \rangle + Cf_\delta \langle \partial_j, \partial_l \rangle \\
&\quad - f_\delta (\langle \mathbf{L} \partial_j, \nabla_{\partial_l} N \rangle + \langle \nabla_{\partial_j} N, \mathbf{L} \partial_l \rangle)
\end{aligned}$$

\square

Proof of Theorem 3.3. Let us define the $(0, 4)$ -tensor B by

$$\begin{aligned}
B_{ijkl} = & -2(\mathbf{L} \wedge \nabla N)_{ijkl} \\
& + \langle \partial_i, (\nabla_{\partial_j} \mathbf{L}) \partial_l \rangle \langle \partial_k, N \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L}) \partial_l \rangle \langle \partial_k, N \rangle \\
& - \langle \partial_i, (\nabla_{\partial_j} \mathbf{L}) \partial_k \rangle \langle \partial_l, N \rangle + \langle \partial_j, (\nabla_{\partial_i} \mathbf{L}) \partial_k \rangle \langle \partial_l, N \rangle \\
& + \langle \partial_k, (\nabla_{\partial_l} \mathbf{L}) \partial_j \rangle \langle \partial_i, N \rangle - \langle \partial_l, (\nabla_{\partial_k} \mathbf{L}) \partial_j \rangle \langle \partial_i, N \rangle \\
& - \langle \partial_k, (\nabla_{\partial_l} \mathbf{L}) \partial_i \rangle \langle \partial_j, N \rangle + \langle \partial_l, (\nabla_{\partial_k} \mathbf{L}) \partial_i \rangle \langle \partial_j, N \rangle
\end{aligned}$$

Clearly the tensor B satisfies $B_{ijkl} = -B_{jikl} = -B_{ijlk}$ and $B_{ijkl} = B_{klij}$, thus inducing a symmetric bilinearform \mathcal{B} on $\Lambda^2(TM)$, $\mathcal{B}(e_i \wedge e_j, e_k \wedge e_l) = B_{ijkl}$ (cf. the discussion in the beginning of this section). In view of Lemmas 3.4 - 3.6 the the claim

$$\mathcal{R}_\delta \approx \mathcal{R} - f_\delta^2 \mathbf{L} \wedge \mathbf{L} + f_\delta \mathcal{B} - 2f'_\delta \mathbf{L} \wedge \mathbf{P}^N + 2f_\delta^2 \mathbf{L}^2 \wedge \mathbf{P}^N + 2Cf_\delta \mathbf{P}^T \wedge \mathbf{P}^N \tag{3.7}$$

follows: Note that since the operators on the right hand side (i.e. their corresponding $(0, 4)$ -tensors) have the same symmetries as the curvature operator, it suffices to consider the following three cases:

Case 1)

Let $i, j, k, l \leq n-1$. Obviously in this case

$$(\mathbf{L} \wedge \mathbf{P}^N)_{ijkl} = (\mathbf{L}^2 \wedge \mathbf{P}^N)_{ijkl} = (\mathbf{P}^T \wedge \mathbf{P}^N)_{ijkl} = 0$$

and $B_{ijkl} = -2(\mathbf{L} \wedge \nabla N)_{ijkl}$. Thus (3.7) follows by Lemma 3.4.

Case 2)

Let $i, j, l \leq n-1$ and $k = n$. Recall that $L_{in} = 0$ for all i and $(P^N)_{in} = 0$ for $i \leq n-1$. Therefore we have

$$(\mathbf{L} \wedge \mathbf{L})_{ijnl} = (\mathbf{L} \wedge \mathbf{P}^N)_{ijnl} = (\mathbf{L}^2 \wedge \mathbf{P}^N)_{ijnl} = (\mathbf{P}^T \wedge \mathbf{P}^N)_{ijnl} = 0$$

Moreover, $(\nabla N)_{in} = \langle \partial_i, \nabla_N N \rangle = 0 = \langle N, \nabla_{\partial_i} N \rangle = (\nabla N)_{ni}$ and therefore

$$B_{ijnl} = \langle \partial_i, (\nabla_{\partial_j} \mathbf{L}) \partial_l \rangle - \langle \partial_j, (\nabla_{\partial_i} \mathbf{L}) \partial_l \rangle$$

and (3.7) follows by Lemma 3.5.

Case 3)

Let $j, l \leq n-1$ and $i = k = n$. Clearly $(\mathbf{L} \wedge \mathbf{L})_{njnl} = 0$. As in case 2) we have $(\mathbf{L} \wedge \nabla N)_{njnl} = 0$ and thus

$$\begin{aligned} B_{njnl} &= \langle N, (\nabla_{\partial_j} \mathbf{L}) \partial_l \rangle - \langle \partial_j, (\nabla_N \mathbf{L}) \partial_l \rangle + \langle N, (\nabla_{\partial_l} \mathbf{L}) \partial_j \rangle - \langle \partial_l, (\nabla_N \mathbf{L}) \partial_j \rangle \\ &= \langle N, (\nabla_{\partial_j} \mathbf{L}) \partial_l \rangle + \langle N, (\nabla_{\partial_l} \mathbf{L}) \partial_j \rangle \end{aligned}$$

where we used that $\nabla_N \mathbf{L} = 0$ (cf. Lemma 2.4). Using the fact that \mathbf{L} is self-adjoint and $\mathbf{L}N = 0$ we compute

$$\begin{aligned} \langle N, (\nabla_{\partial_j} \mathbf{L}) \partial_l \rangle &= \langle N, \nabla_{\partial_j} (\mathbf{L} \partial_l) \rangle - \langle N, \mathbf{L} (\nabla_{\partial_j} \partial_l) \rangle \\ &= \partial_j \langle N, \mathbf{L} \partial_l \rangle - \langle \nabla_{\partial_j} N, \mathbf{L} \partial_l \rangle \\ &= -\langle \nabla_{\partial_j} N, \mathbf{L} \partial_l \rangle \end{aligned}$$

which gives us

$$B_{njnl} = -(\langle \nabla_{\partial_j} N, \mathbf{L} \partial_l \rangle) + \langle \nabla_{\partial_l} N, \mathbf{L} \partial_j \rangle$$

(3.7) follows by Lemma 3.6 and we are done.

4 The Riemannian curvature operator of $g^{1,'}$

Recall that $g^{1,'}$ is the extension of g^1 on a small neighborhood of Γ in M_0 , as introduced in Lemma 2.2. In this section we compare the Riemannian curvature operators on Γ with respect to the metrics g and $g^{1,'}$ (cf. [2], § 9).

We define the selfadjoint operator \mathbf{G}_1 on TM_0 by $\langle \cdot, \mathbf{G}_1 \cdot \rangle = \langle \cdot, \cdot \rangle'_1$.

Proposition 4.1. *Let \mathcal{R}'_1 be the Riemannian curvature operator with respect to $g^{1,'}$. On Γ we have*

$$\mathcal{R}'_1 = \mathcal{R} - \mathcal{A} + \mathcal{B} + 2\mathcal{L}^2 - \nabla_N^2 \mathcal{G}_1$$

where \mathcal{A} , \mathcal{B} and \mathcal{L}^2 are as in Theorem 3.3 and $\nabla_N^2 \mathcal{G}_1 := (\nabla_N^2 \mathbf{G}_1) \wedge \mathbf{P}^N$.

In particular, since $\mathcal{R}'_1 = \mathcal{R}_1$ holds on Γ independently of the extension g'_1 , and $\mathcal{R}_1 \geq \kappa$ by assumption, we have

$$\mathcal{R} - \mathcal{A} + \mathcal{B} + 2\mathcal{L}^2 - \nabla_N^2 \mathcal{G}_1 \geq \kappa \mathcal{I} \quad (4.1)$$

on Γ , which is an estimate we will use in the next section.

Proof. As in [2], Lemma 9.1, we have to check the approximate identities for \mathbf{G}_1 which correspond with the ones in (2.3). For the convenience of the reader we repeat the computations from [2]. Let $X, Y \in \{\partial_1, \dots, \partial_{n-1}\}$. On Γ we have

$$\mathbf{G}_1 = \mathbf{I}$$

which implies

$$\nabla_X \mathbf{G}_1 = 0$$

on Γ . Moreover, we have $\nabla_N \mathbf{G}_1 = 2\mathbf{L}$ on Γ . Indeed

$$\begin{aligned}
\langle X, (\nabla_N \mathbf{G}_1)Y \rangle &= \langle X, \nabla_N(\mathbf{G}_1 Y) \rangle - \langle X, \mathbf{G}_1(\nabla_N Y) \rangle \\
&\stackrel{\text{on } \Gamma}{=} N\langle X, \mathbf{G}_1 Y \rangle - \langle \nabla_N X, Y \rangle - \langle X, \nabla_N Y \rangle \\
&= N\langle X, Y \rangle'_1 - \langle \nabla_N X, Y \rangle - \langle X, \nabla_N Y \rangle \\
&= \langle \nabla^{1, \prime}_N X, Y \rangle'_1 + \langle X, \nabla^{1, \prime}_N Y \rangle'_1 - \langle \nabla_N X, Y \rangle - \langle X, \nabla_N Y \rangle \\
&= 2L_0(X, Y) - 2L_1(X, Y) \\
&= \langle X, 2\mathbf{L}Y \rangle
\end{aligned}$$

where we used that in our coordinates the second fundamental forms of Γ with respect to N are $-\langle X, \nabla_N Y \rangle$ and $\langle X, \nabla^{1, \prime}_N Y \rangle^{1, \prime}$ (cf. Remark 2.8). Furthermore, recall that by construction we have $\mathbf{G}_1 N = N$ (cf. Lemma 2.2) and therefore

$$(\nabla_N \mathbf{G}_1)N = \nabla_N(\mathbf{G}_1 N) - \mathbf{G}_1(\nabla_N N) = 0 = \mathbf{L}N$$

Finally, $\nabla_N \mathbf{G}_1 = 2\mathbf{L}$ implies $\nabla_X \nabla_N \mathbf{G}_1 = 2\nabla_X \mathbf{L}$ on Γ .

We can now repeat the computations from the previous section, where the only difference occurs due to the $\nabla_N \nabla_N \mathbf{G}_1$ term. \square

5 Estimating \mathcal{R}_δ on M_0

The goal of this section is to show that $\mathcal{R}_\delta \geq (\kappa - \varepsilon(\delta))\mathcal{I}_\delta$ holds on M_0 .

Lemma 5.1. *We have*

$$\mathcal{R} - f_\delta^2 \mathcal{A} + f_\delta \mathcal{B} \geq (\kappa - \varepsilon(\delta))\mathcal{I} + 2f_\delta(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2 \mathcal{G}_1) \quad (5.1)$$

where $\varepsilon(\delta)$ tends to zero as $\delta \rightarrow 0$.

Proof. Since Γ is compact it suffices to show that

$$\mathcal{R}(\alpha, \alpha) - f_\delta^2 \mathcal{A}(\alpha, \alpha) + f_\delta \mathcal{B}(\alpha, \alpha) \geq \kappa \mathcal{I}(\alpha, \alpha) + 2f_\delta(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2 \mathcal{G}_1)(\alpha, \alpha) - \varepsilon(\delta)\mathcal{I}(\alpha, \alpha)$$

holds on a small neighborhood U of a point $p \in \Gamma$ for every 2-form α on U where $\varepsilon(\delta)$ does not depend on α . Let us fix a coordinate neighborhood (U, φ) of p as in Section 2. W.l.o.g. we assume that α has fixed coefficients satisfying $\sum_{i,j=1}^n (\alpha^{ij})^2 = 1$.

We proceed as in Lemma 9.2 of [2]. Off a δ -neighborhood of Γ , i.e. $f_\delta(x^n) = 0$ the inequality holds without an error term. For $f_\delta(x^n) = 1$, i.e. on Γ the inequality follows from (4.1) and the assumption $\mathcal{R}_1 \geq \kappa$.

Let us now fix a point $\hat{x} = (x^1, \dots, x^{n-1}) \in U \cap \Gamma$ and look at the inequality on the line segment $\{(\hat{x}, x^n) : x^n \in [0, \delta]\}$. Let

$$\mathcal{Q} = -\mathcal{L}^2 + \frac{1}{2}\nabla_N^2 \mathcal{G}_1$$

For $x^n \in [0, \delta^4]$ we have $f_\delta(x^n) \in [0, 1]$ (cf. the definition of f_δ). If the quantities $\mathcal{R}(\alpha, \alpha)$, $\mathcal{A}(\alpha, \alpha)$, $\mathcal{B}(\alpha, \alpha)$ and $\mathcal{Q}(\alpha, \alpha)$ would not depend x^n , the inequality

$$\mathcal{R}(\alpha, \alpha) - f_\delta^2 \mathcal{A}(\alpha, \alpha) + f_\delta \mathcal{B}(\alpha, \alpha) \geq \kappa \mathcal{I}(\alpha, \alpha) + 4f_\delta \mathcal{Q}(\alpha, \alpha) \quad (5.2)$$

would hold without an error term since it holds for $f_\delta = 0$ and $f_\delta = 1$ and the function

$$\begin{aligned}
[0, 1] &\rightarrow \mathbb{R} \\
y &\mapsto \mathcal{R}(\alpha, \alpha) - y^2 \mathcal{A}(\alpha, \alpha) + y \mathcal{B}(\alpha, \alpha)
\end{aligned}$$

is concave (note that $\mathbf{L} \geq 0$ implies $\mathcal{A} = \mathbf{L} \wedge \mathbf{L} \geq 0$ by Lemma 3.2). However $\mathcal{R}(\alpha, \alpha)$, $\mathcal{A}(\alpha, \alpha)$, $\mathcal{B}(\alpha, \alpha)$ and $\mathcal{Q}(\alpha, \alpha)$ do depend on x^n , but they are smooth on M_0 and hence almost constant for small x^n . Indeed, one has for instance

$$\begin{aligned} |\mathcal{R}(\alpha, \alpha)(\hat{x}, s) - \mathcal{R}(\alpha, \alpha)(\hat{x}, t)| &= \frac{1}{4} |R_{ijkl}(\hat{x}, s) - R_{ijkl}(\hat{x}, t)| |\alpha^{ij} \alpha^{kl}| \\ &\leq \delta c(n) \sup_{i,j,k,l} \|R_{ijkl}\|_{C^1(U)} \end{aligned}$$

for all $s, t \in [0, \delta]$, which tends to zero since the C^1 -norm of the coordinate functions is bounded if we choose U small enough. Therefore (5.2) holds up to a small error term $\varepsilon(\delta)$ on the right hand side for $x^n \in [0, \delta^4]$.

For $x^n \in [\delta^4, \delta]$ we have $f_\delta(x^n) \in [-\delta^2, 0]$. $\mathcal{A}, \mathcal{B}, \mathcal{I}$ and \mathcal{Q} are uniformly bounded near Γ , therefore (5.2) holds for all $x^n \in [0, \delta]$ if we choose δ sufficiently small and subtract another $\varepsilon(\delta)$ on the right hand side. □

Proposition 5.2 (cf. [2], Lemma 10.1). *If the constant C in the definition of g_δ is chosen large enough, then for small $\delta > 0$*

$$\mathcal{R}_\delta \geq (\kappa - \varepsilon(\delta)) \mathcal{I}_\delta$$

where $\varepsilon(\delta) \rightarrow 0$ as δ tends to zero.

Proof. Since $g_\delta \rightarrow g$ in the C^0 -sense, it suffices to show

$$\mathcal{R}_\delta \geq (\kappa - \varepsilon(\delta)) \mathcal{I}$$

From Proposition 3.3 and Lemma 5.1 we get

$$\begin{aligned} \mathcal{R}_\delta &\approx \mathcal{R} - f_\delta^2 \mathcal{A} + f_\delta \mathcal{B} - 2f'_\delta \mathcal{L} + 2f_\delta^2 \mathcal{L}^2 + 2C f_\delta \hat{\mathcal{I}} \\ &\geq \kappa \mathcal{I} + 2f_\delta (-\mathcal{L}^2 + \frac{1}{2} \nabla_N^2 \mathcal{G}_1 + C \hat{\mathcal{I}}) - 2f'_\delta \mathcal{L} + 2f_\delta^2 \mathcal{L}^2 - \varepsilon(\delta) \mathcal{I} \end{aligned}$$

By definition we have

$$-\mathcal{L}^2 + \frac{1}{2} \nabla_N^2 \mathcal{G}_1 + C \hat{\mathcal{I}} = (-\mathbf{L}^2 + \frac{1}{2} \nabla_N^2 \mathbf{G}_1 + C \mathbf{P}^T) \wedge \mathbf{P}^N \quad (5.3)$$

Note that the operators \mathbf{L}^2 and $\nabla_N^2 \mathbf{G}_1$ vanish on $T\Gamma(d)^\perp$. Therefore (5.3) becomes nonnegative near Γ for a large enough C in view of Lemma 3.2 (recall that \mathbf{P}^N is nonnegative). Moreover, $-\mathcal{L}^2 + \frac{1}{2} \nabla_N^2 \mathcal{G}_1 + C \hat{\mathcal{I}}$ is uniformly bounded near Γ , and $f_\delta \geq -\delta^2$ by definition. Therefore

$$2f_\delta (-\mathcal{L}^2 + \frac{1}{2} \nabla_N^2 \mathcal{G}_1 + C \hat{\mathcal{I}}) \geq -\varepsilon(\delta) \mathcal{I}$$

f'_δ is negative on $[0, \delta^4]$, does not exceed $\varepsilon(\delta)$ on $[\delta^4, \delta]$ and vanishes else. $\mathcal{L} = \mathbf{L} \wedge \mathbf{P}^N$ is nonnegative and uniformly bounded near Γ , which gives us $-4f'_\delta \mathcal{L} \geq -\varepsilon(\delta) \mathcal{I}$. Obviously $f_\delta^2 \mathcal{L}^2 = f_\delta^2 \mathbf{L}^2 \wedge \mathbf{P}^N$ is nonnegative, and we are done. □

Corollary 5.3. *The weakly defined Riemannian curvature operator of the $W_{loc}^{2,\infty}$ -metric $g_{(\delta)}$ on M , (recall that $g_{(\delta)}|_{M_0} = g_\delta$ and $g_{(\delta)}|_{M_1} = g_1$) satisfies $\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta)$ a.e. (everywhere except on Γ).*

Proof. In local coordinates the Riemannian curvature tensor of some metric h is given by

$$R(h)_{ijkl} = \partial_j \partial_k h_{il} + \partial_i \partial_l h_{jk} - \partial_j \partial_l h_{ik} - \partial_i \partial_k h_{jl} + (h^{-1} \bullet \partial h \bullet \partial h)_{ijkl} \quad (5.4)$$

where \bullet means contracting tensors using the metric. Since the second derivatives enter (5.4) linearly, $\mathcal{R}(g_{(\delta)})$ can be defined on M in the weak sense. $\mathcal{R}(g_{(\delta)}) \geq \kappa - \varepsilon(\delta)$ a.e. follows from Proposition 5.2 and the assumption $\mathcal{R}(g_1) \geq \kappa$. □

6 Mollifying $g_{(\delta)}$

By mollifying $g_{(\delta)}$ we construct a family of smooth metrics with properties as required in Definition 1.1.

Proposition 6.1. *There exists a family of smooth metrics $\tilde{g}_{(\delta)}$ such that*

$$\tilde{g}_{(\delta)} \rightarrow g \quad \text{uniformly on compact subsets of } M$$

and

$$\tilde{\mathcal{R}}_{(\delta)} \geq (\kappa - \tilde{\varepsilon}(\delta))\tilde{\mathcal{I}}_{(\delta)}$$

holds with $\tilde{\varepsilon}(\delta) \rightarrow 0$.

Proof. Let us fix a small $\delta > 0$. We choose a locally finite cover of coordinate neighborhoods (U_s) such that $U_s \subset \subset U'_s$ for some coordinate chart U'_s . Since Γ is compact, we may assume w.l.o.g. that $U'_s \cap \Gamma = \emptyset$ for $s > N$ for some $N \in \mathbb{N}$. We denote the coordinate functions of $g_{(\delta)}$ on U'_s by $(g_{(\delta)}^s)_{ij}$. After choosing U'_s even smaller if necessary we may also assume that $\|(g_{(\delta)}^s)_{ij}\|_{C^1(U'_s)} \leq C < \infty$ for all $s \leq N$. For $s \leq N$ and $x \in U_s$ let

$$(g_{(\delta)}^{s,h})_{ij}(x) = (\rho_h * (g_{(\delta)}^s)_{ij})(x) = \int_{|z| \leq 1} \rho(z) (g_{(\delta)}^s)_{ij}(x - hz) dz \quad (6.1)$$

where $\rho \in C_0^\infty(\mathbb{R}^n)$ satisfies $\text{supp } \rho \subset B_1(0)$ and $\int_{\mathbb{R}^n} \rho = 1$, and h is small enough so that for all $s \leq N$ $x - hz$ lies in U'_s for all $z \in B_1(0)$ (here we identified the coordinate neighborhoods on M with the corresponding neighborhoods on \mathbb{R}^n). $g_{(\delta)}^{s,h}$ is a well defined metric on U_s which converges to $g_{(\delta)}|_{U_s}$ in the C^1 -sense. Let (η_s) be a partition of unity on M such that $\text{supp } \eta_s \subset U_s$ for all s . For h as above we then define a smooth metric $g_{(\delta)}^h$ on M by

$$g_{(\delta)}^h = \sum_{s \leq N} \eta_s g_{(\delta)}^{s,h} + \sum_{s > N} \eta_s g_{(\delta)} \quad (6.2)$$

We now calculate the Riemannian curvature tensor $R(g_{(\delta)}^h)$ using the formula (5.4). The terms which do not involve any derivatives of the unity functions η_s give us just the mollified Riemannian curvature tensor $(R(g_{(\delta)}))^h$, constructed in the same way as $g_{(\delta)}^h$ in (6.1) and (6.2). The other terms vanish uniformly on M as h tends to zero. We shall verify this exemplarily for one of them. After fixing a coordinate chart (U, φ) we compute

$$\begin{aligned} & \left| \sum_{s \leq N} \partial_j \partial_k \eta_s (g_{(\delta)}^{s,h})_{il} + \sum_{s > N} \partial_j \partial_k \eta_s (g_{(\delta)})_{il} \right| \\ &= \left| \sum_{s \leq N} \partial_j \partial_k \eta_s (g_{(\delta)})_{il} + \sum_{s \leq N} \partial_j \partial_k \eta_s ((g_{(\delta)}^{s,h})_{il} - (g_{(\delta)})_{il}) + \sum_{s > N} \partial_j \partial_k \eta_s (g_{(\delta)})_{il} \right| \\ &\leq \underbrace{|\partial_j \partial_k (\sum_{s \geq 1} \eta_s)(g_{(\delta)})_{il}|}_{\equiv 1} + \sum_{s \leq N} |\partial_j \partial_k \eta_s| |(g_{(\delta)}^{s,h})_{il} - (g_{(\delta)})_{il}| \\ &\leq N \left(\max_{s=1, \dots, N} \|\eta_s\|_{C^2(U_s)} \right) \left(\max_{s=1, \dots, N} \max_{i, l=1, \dots, n} \|(g_{(\delta)}^{s,h})_{il} - (g_{(\delta)})_{il}\|_{C^0(U_s)} \right) \\ &\xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

All in all we obtain

$$|(R(g_{(\delta)}^h))_{ijkl} - (R(g_{(\delta)}))^h_{ijkl}| \leq \varepsilon(\delta, h)$$

where $\varepsilon(\delta, h) \xrightarrow{h \rightarrow 0} 0$ for every fixed δ , which implies that

$$\mathcal{R}(g_{(\delta)}^h) \geq (\mathcal{R}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta, h) \mathcal{I}(g_{(\delta)}) \quad (6.3)$$

Moreover, Corollary 5.3 implies

$$(\mathcal{R}(g_{(\delta)}))^h \geq (\kappa - \varepsilon(\delta))(\mathcal{I}(g_{(\delta)}))^h \quad (6.4)$$

holds. Indeed, for any two form α on $U_{s'}$ (w.l.o.g. with fixed coefficients) we have

$$\begin{aligned} (\mathcal{R}(g_{(\delta)}))^{s,h}(x)(\alpha, \alpha) &= \int_{|z| \leq 1} \rho(z) \frac{1}{4} (R(g_{(\delta)}))_{ijkl}^s (x - hz) \alpha^{ij} \alpha^{kl} dz \\ &\stackrel{5.3}{\geq} (\kappa - \varepsilon(\delta)) \int_{|z| \leq 1} \rho(z) \frac{1}{4} (\mathcal{I}(g_{(\delta)}))_{ijkl}^s (x - hz) \alpha^{ij} \alpha^{kl} dz \\ &= (\kappa - \varepsilon(\delta)) (\mathcal{I}(g_{(\delta)}))^{s,h}(x)(\alpha, \alpha) \end{aligned}$$

Combining (6.3) and (6.4) we arrive at

$$\begin{aligned} \mathcal{R}(g_{(\delta)}^h) &\geq (\kappa - \varepsilon(\delta)) (\mathcal{I}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta, h) \mathcal{I}(g_{(\delta)}) \\ &\geq (\kappa - \varepsilon(\delta)) (1 \pm \varepsilon(\delta)) \mathcal{I}(g_{(\delta)}^h) - \tilde{\varepsilon}(\delta, h) (1 + \varepsilon(\delta)) \mathcal{I}(g_{(\delta)}^h) \end{aligned}$$

where we used the fact that for every fixed δ both $(\mathcal{I}(g_{(\delta)}))^h$ and $\mathcal{I}(g_{(\delta)}^h)$ approach $\mathcal{I}(g_{(\delta)})$ as h tends to zero (\pm refers to $\kappa \geq 0, \kappa < 0$, respectively). Since $\tilde{\varepsilon}(\delta, h) \rightarrow 0$ as $h \rightarrow 0$ for every fixed δ , we may choose h small enough such that $\tilde{\varepsilon}(\delta, h) \leq \varepsilon(\delta)$, thereby obtaining

$$\mathcal{R}(g_{(\delta)}^h) \geq (\kappa - (|\kappa| + 3)\varepsilon(\delta)) \mathcal{I}(g_{(\delta)}^h)$$

and the desired result follows with $\tilde{g}_{(\delta)} = g_{(\delta)}^h$ and $\tilde{\varepsilon}(\delta) = (|\kappa| + 3)\varepsilon(\delta)$. \square

7 A similar result for other operators

As mentioned in the introduction, an analogue result can be shown for manifolds with lower bounds on the Ricci curvature, scalar curvature, bi-curvature, isotropic curvature and flag curvature, respectively.

7.1 Manifolds with Ricci curvature $\geq \kappa$

Theorem 7.1. *Let (M_0, g_0) , (M_1, g_1) , (M, g) and $L = L_0 + L_1$ be as in Theorem 1.2. Suppose that $\text{Ric}(g_0)$ and $\text{Ric}(g_1)$ are at least κ (in the sense of eigenvalues). If L is positive semidefinite, then $\text{Ric}(g)$ is at least κ (in a similar sense as in Definition 1.1).*

Proof. Given a symmetric bilinear form \mathcal{T} on $\Lambda^2(TM)$ and a metric h we denote

$$\text{Ric}_h(\mathcal{T}) = h^{jl} T(\cdot, \partial_j, \cdot, \partial_l)$$

where $T(\partial_i, \partial_j, \partial_k, \partial_l) = \mathcal{T}(\partial_i \wedge \partial_j, \partial_k \wedge \partial_l)$. The strategy of the proof is similar to that of the proof of Theorem 1.2. We show

- (a) The curvature operator of the modified metric g_δ on M_0 satisfies $\text{Ric}_{g_\delta}(\mathcal{R}_\delta) \geq (\kappa - \varepsilon(\delta))g_\delta$ with $\varepsilon(\delta) \rightarrow 0$ (this corresponds with Lemma 5.2)
- (b) By mollifying $g_{(\delta)}$ we construct a family of smooth metrics which approximate g in the C^0 -sense and have Ricci curvature at least $\kappa - \varepsilon(\delta)$.

(a): Here we may simplify the argument of the previous sections. Recall that we identify endomorphisms and bilinear forms on TM_0 in the sense of Notation 2.3. In view of this identification, we have $g = \text{id}_{TM_0}$. Since $g_\delta \approx g$ on M_0 , it suffices to show $\text{Ric}_{g_\delta}(\mathcal{R}_\delta) \geq (\kappa - \varepsilon(\delta))\text{id}_{TM_0}$. By (3.1) we have

$$\begin{aligned} \text{Ric}_{g_\delta}(\mathcal{R}_\delta) &\geq \text{Ric}_{g_\delta}(\mathcal{R}) - f_\delta^2 \text{Ric}_{g_\delta}(\mathcal{A}) + f_\delta \text{Ric}_{g_\delta}(\mathcal{B}) \\ &\quad - 2f'_\delta \text{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \text{Ric}_{g_\delta}(\mathcal{L}^2) + 2Cf_\delta \text{Ric}_{g_\delta}(\hat{\mathcal{T}}) - \varepsilon(\delta)\text{id}_{TM_0} \end{aligned}$$

Since f_δ is bounded and $g_\delta \rightarrow g$ in the C^0 -sense, we may replace Ric_{g_δ} by Ric_g everywhere except in the f'_δ term, i.e. we have

$$\begin{aligned} \text{Ric}_{g_\delta}(\mathcal{R}_\delta) &\geq \text{Ric}_g(\mathcal{R}) - f_\delta^2 \text{Ric}_g(\mathcal{A}) + f_\delta \text{Ric}_g(\mathcal{B}) \\ &\quad - 2f'_\delta \text{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \text{Ric}_g(\mathcal{L}^2) + 2Cf_\delta \text{Ric}_g(\hat{\mathcal{T}}) - \varepsilon(\delta)\text{id}_{TM_0} \end{aligned} \tag{7.1}$$

Recall that $\hat{\mathcal{L}} = \mathbf{P}^T \wedge \mathbf{P}^N$ (cf. Notation 2.6). We compute

$$\begin{aligned}
(\text{Ric}_g(\mathbf{P}^T \wedge \mathbf{P}^N))_{ik} &= \frac{1}{2} g^{jl} (P_{ik}^T P_{jl}^N - P_{jk}^T P_{il}^N + P_{ik}^N P_{jl}^T - P_{jk}^N P_{il}^T) \\
&= \frac{1}{2} (\text{tr}_g(\mathbf{P}^N) P_{jk}^T + \text{tr}_g(\mathbf{P}^T) P_{jk}^N) \\
&= \frac{1}{2} (P_{ik}^T + (n-1) P_{ik}^N)
\end{aligned} \tag{7.2}$$

If we assume that $n \geq 2$ (the case $n = 1$ is trivial), this implies

$$\text{Ric}_g(\hat{\mathcal{L}}) \geq \frac{1}{2} (\mathbf{P}^T + \mathbf{P}^N) = \frac{1}{2} \text{id}_{TM_0} \tag{7.3}$$

Therefore, using the assumption $\text{Ric}_g(\mathcal{R}) \geq \kappa$, we can estimate the right hand side of (7.1) from below by

$$\begin{aligned}
\text{Ric}_{g_\delta}(\mathcal{R}_\delta) &\geq (\kappa - \varepsilon(\delta)) \text{id}_{TM_0} - f_\delta^2 \text{Ric}_g(\mathcal{A}) + f_\delta \text{Ric}_g(\mathcal{B}) \\
&\quad - 2f_\delta' \text{Ric}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \text{Ric}_g(\mathcal{L}^2) + C f_\delta \text{id}_{TM_0} \\
&= (\kappa - \varepsilon(\delta)) \text{id}_{TM_0} - 2f_\delta' \text{Ric}_{g_\delta}(\mathcal{L}) \\
&\quad + f_\delta (-f_\delta \text{Ric}_g(\mathcal{A}) + \text{Ric}_g(\mathcal{B}) + 2f_\delta \text{Ric}_g(\mathcal{L}^2) + C \text{id}_{TM_0})
\end{aligned} \tag{7.4}$$

The operators \mathcal{A} , \mathcal{B} and \mathcal{L}^2 are smooth and hence uniformly bounded near Γ . Therefore, the term in parenthesis in (7.4) is nonnegative for large enough fixed C and bounded from above¹. Since $f_\delta \in [-\delta^2, 1]$, the last expression in (7.4) is $\geq -\varepsilon(\delta) \text{id}_{TM_0}$, and we arrive at

$$\text{Ric}_{g_\delta}(\mathcal{R}_\delta) \geq (\kappa - \varepsilon(\delta)) \text{id}_{TM_0} - 2f_\delta' \text{Ric}_{g_\delta}(\mathcal{L}) \tag{7.5}$$

Finally, we compute the f_δ' -term in (7.5). Let us fix a point $x \in M_0$ near Γ . In the construction of local coordinates in Section 2 we may additionally choose x^1, \dots, x^{n-1} such that $\partial_1(x), \dots, \partial_{n-1}(x)$ are orthonormal with respect to $g(x)$ and $L(x)$ is diagonal. By construction this implies that $g_\delta(x)$ is diagonal, $(g_\delta)_{jl}(x) = \mu_l \delta_{jl}$, where $\mu_l > 0$ since g_δ is positive definite. Moreover, we still have $(P^N)_{ij} = \delta_{in} \delta_{jn}$ in these coordinates. Therefore, given a vector ξ , in x we compute using $L_{kn} = 0$ for $k = 1, \dots, n$:

$$\begin{aligned}
(\text{Ric}_{g_\delta}(\mathcal{L}))(\xi, \xi) &= g_\delta^{jl} (\mathbf{L} \wedge \mathbf{P}^N)_{ijkl} \xi^i \xi^k = \sum_{l=1}^n \frac{1}{\mu_l} (\mathbf{L} \wedge \mathbf{P}^N)_{ilkl} \xi^i \xi^k \\
&= \frac{1}{2} \sum_{l=1}^n \frac{1}{\mu_l} (L_{ik} P_{ll}^N - L_{lk} P_{il}^N + P_{ik}^N L_{ll} - P_{lk}^N L_{il}) \xi^i \xi^k \\
&= \frac{1}{2} \frac{1}{\mu_n} L(\xi, \xi) + \frac{1}{2} (\xi^n)^2 \sum_{l=1}^n \frac{1}{\mu_l} L_{ll} \geq 0
\end{aligned}$$

since $L \geq 0$ by assumption. We then proceed as in the proof of Proposition 5.2 and estimate the f_δ' -term from below by $-\varepsilon(\delta) \text{id}_{TM_0}$.

(b): Let us fix a $\delta > 0$. We construct the metrics $g_{(\delta)}^h$ as in Section 6. By (6.3) and since $g_{(\delta)}^h \rightarrow g_{(\delta)}$ uniformly as $h \rightarrow 0$ we have

$$\text{Ric}_{g_{(\delta)}^h}(\mathcal{R}(g_{(\delta)}^h)) \geq \text{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^h - \tilde{\varepsilon}(\delta, h)(g_{(\delta)})$$

¹ Note that at this point we simplified the argument of Section 5. $\text{Ric}_g(\mathbf{P}^T \wedge \mathbf{P}^N)$ is estimated from below by the positive definite operator $\frac{1}{2} \text{id}_{TM_0}$, hence the \mathcal{A}, \mathcal{B} and \mathcal{L}^2 terms are absorbed by $C \text{id}_{TM_0}$. When considering the full curvature tensor, the corresponding operator $C \mathbf{P}^T \wedge \mathbf{P}^N$ has nontrivial kernel, which is why we needed the concavity argument of Lemma 5.1.

where $\tilde{\varepsilon}(\delta, h) \rightarrow 0$ as $h \rightarrow 0$. Given a vectorfield X on $U_{s'}$ which has constant coefficients not exceeding 1, on U_s we compute using (a) and the mean value theorem

$$\begin{aligned}
\text{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^{h,s}(x)(X, X) &= \int_{|z| \leq 1} \rho(z)(g_{(\delta)})^{jl}(x)(R(g_{(\delta)}))_{ijkl}^s(x - hz)X^iX^k dz \\
&= \int_{|z| \leq 1} \rho(z)(g_{(\delta)})^{jl}(x - hz)(R(g_{(\delta)}))_{ijkl}^s(x - hz)X^iX^k dz \\
&+ h \int_{|z| \leq 1} \rho(z)D(g_{(\delta)})^{jl}(\xi_{x,hz})z(R(g_{(\delta)}))_{ijkl}^s(x - hz)X^iX^k dz \\
&\geq (\kappa - \varepsilon(\delta))g_{(\delta)}^{s,h}(X, X) - hC(\delta) \\
&\geq (\kappa - 2\varepsilon(\delta))g_{(\delta)}^{s,h}(X, X)
\end{aligned}$$

where $\xi_{x,hz} = (1 - t)x + thz$ for some $t \in [0, 1]$, and $C(\delta)$ depends on the bound of $\mathcal{R}(g_{(\delta)})$ near Γ , which is finite for every fixed δ . Note that for every fixed δ we may choose h small enough so that $hC(\delta) \leq \varepsilon(\delta)$. Since $U_s \cap \Gamma \neq \emptyset$ only for finitely many s , we deduce

$$\text{Ric}_{g_{(\delta)}}(\mathcal{R}(g_{(\delta)}))^h \geq (\kappa - 2\varepsilon(\delta))g_{(\delta)}^h$$

Thus

$$\text{Ric}_{g_{(\delta)}^h}(\mathcal{R}(g_{(\delta)}^h)) \geq (\kappa - 2\varepsilon(\delta))g_{(\delta)}^h - \tilde{\varepsilon}(\delta, h)(g_{(\delta)})$$

Finally we choose h even smaller such that $\tilde{\varepsilon}(\delta, h) \leq \varepsilon(\delta)$ and $g_{(\delta)} \leq (1 + \varepsilon(\delta))g_{(\delta)}^h$, and the result follows with $\tilde{g}_{(\delta)} = g_{(\delta)}^h$ and $\tilde{\varepsilon}(\delta) = 4\varepsilon(\delta)$. \square

7.2 Manifolds with scalar curvature $\geq \kappa$

The scalar curvature of a C^2 -smooth Riemannian metric g is defined as $\text{sc}(g) = \text{tr}_g \text{Ric}_g = g^{ik}g^{jl}R_{ijkl}^g$. As mentioned in the introduction, in the scalar curvature case we may weaken the assumption $L \geq 0$ on Γ to $\text{tr}_g L \geq 0$ on Γ , i.e. the sum of the mean curvatures of g_0 and g_1 is nonnegative.

Theorem 7.2. *Let (M_0, g_0) , (M_1, g_1) , (M, g) and $L = L_0 + L_1$ be as in Theorem 1.2. Suppose that $\text{sc}(g_0)$ and $\text{sc}(g_1)$ are at least κ . If $\text{tr}_g L \geq 0$ on Γ , then $\text{sc}(g) \geq \kappa$, (in a similar sense as in Definition 1.1).*

Proof. First let us assume $\text{tr}_g L > 0$ on Γ . In analogy to Lemma 2.4, we need to verify that the extension of L satisfies $\text{tr}_g L > 0$, if so does the initial operator on Γ . In fact, for $x \in M_0$ near Γ we have $\text{tr}_{g(x)} L(x) = \text{tr}_{g(\hat{x})} L(\hat{x})$, where \hat{x} is the point of Γ nearest to x . Indeed, let $x \in M_0$ be a point near Γ such that the extension \mathbf{L} is well defined in x . Recall that for $X \in T_x M_0$ we defined $\mathbf{L}X = P^{-1}\mathbf{L}PX$, where P is the parallel transportation along the integral curves of the normal field N , which takes $X \in T_x M_0$ to $PX \in T_{\hat{x}} M_0$. Let e_1, \dots, e_n be an orthonormal basis of $T_x M_0$, and let $g_{ij}(x) = \langle e_i, e_j \rangle_{g(x)} = \delta_{ij}$ and $L_{ij}(x) = \langle \mathbf{L}(x)e_i, e_j \rangle_{g(x)}$. We compute

$$\begin{aligned}
\text{tr}_{g(x)} L(x) &= g^{ij}(x)L_{ij}(x) = \sum_{i=1}^n \langle \mathbf{L}(x)e_i, e_i \rangle_{g(x)} \\
&= \sum_{i=1}^n \langle P^{-1}\mathbf{L}(\hat{x})Pe_i, e_i \rangle_{g(x)} = \sum_{i=1}^n \langle \mathbf{L}(\hat{x})Pe_i, Pe_i \rangle_{g(\hat{x})} \\
&= \sum_{i=1}^n L(\hat{x})(Pe_i, Pe_i) = \text{tr}_{g(\hat{x})} L(\hat{x})
\end{aligned} \tag{7.6}$$

since Pe_1, \dots, Pe_n is an orthonormal basis of $T_{\hat{x}} M_0$.

Given a metric h and a bilinear form $\mathcal{T} \in \Lambda^2(TM)$ we denote:

$$\text{sc}_h(\mathcal{T}) = h^{ik}h^{kl}\mathcal{T}_{ijkl}$$

where $T_{ijkl} = \mathcal{T}(\partial_i \wedge \partial_j, \partial_k \wedge \partial_l)$. By (3.1) we have

$$\begin{aligned}
\text{sc}_{g_\delta}(\mathcal{R}_\delta) &\geq \text{sc}_{g_\delta}(\mathcal{R}) - f_\delta^2 \text{sc}_{g_\delta}(\mathcal{A}) + f_\delta \text{sc}_{g_\delta}(\mathcal{B}) \\
&\quad - 2f'_\delta \text{sc}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \text{sc}_{g_\delta}(\mathcal{L}^2) + 2C f_\delta \text{sc}_{g_\delta}(\hat{\mathcal{I}}) - \varepsilon(\delta) \\
&\geq \text{sc}_g(\mathcal{R}) - f_\delta^2 \text{sc}_g(\mathcal{A}) + f_\delta \text{sc}_g(\mathcal{B}) \\
&\quad - 2f'_\delta \text{sc}_{g_\delta}(\mathcal{L}) + 2f_\delta^2 \text{sc}_g(\mathcal{L}^2) + 2C f_\delta \text{sc}_g(\hat{\mathcal{I}}) - \varepsilon(\delta)
\end{aligned} \tag{7.7}$$

where we used that $g_\delta \rightarrow g$ in the C^0 -sense and the fact that all terms, except for the f'_δ -term, remain bounded as $\delta \rightarrow 0$. In view of (7.2) we have

$$\text{sc}_g \hat{\mathcal{I}} = \frac{1}{2} g^{ik} (P_{ik}^T + (n-1)P_{ik}^N) = n-1 > 0$$

if $n \geq 2$. Similarly as in the previous section, we use the assumption $\text{sc}_g(\mathcal{R}) \geq 0$ and the fact that \mathcal{A} , \mathcal{B} and \mathcal{L}^2 are bounded near Γ and f_δ is almost nonnegative, so that after choosing C large enough, we may estimate (7.7) from below by

$$\text{sc}_{g_\delta}(\mathcal{R}_\delta) \geq \kappa - \varepsilon(\delta) - 2f'_\delta \text{sc}_{g_\delta}(\mathcal{L}) \tag{7.8}$$

Consider the f'_δ -term in the above expression. As in the previous section, in some point $x \in M_0$ near Γ we may choose local coordinates such that $g_{ij} = \delta_{ij}$, $L_{ij} = \lambda_i \delta_{ij}$, $(g_\delta)_{ij} = \mu_i^\delta \delta_{ij}$ and $P_{ij}^N = \delta_{in} \delta_{jn}$. In these coordinates we have (recall that $\lambda_n = L_{nn} = 0$ and $\mu_n^\delta = 1$)

$$\begin{aligned}
(\text{sc}_{g_\delta}(\mathcal{L})) &= g_\delta^{ik} g_\delta^{jl} (\mathbf{L} \wedge \mathbf{P}^N)_{ijkl} = \sum_{i,j=1}^n \frac{1}{\mu_i^\delta} \frac{1}{\mu_j^\delta} (\mathbf{L} \wedge \mathbf{P}^N)_{ijij} \\
&= \frac{1}{2} \sum_{i,j=1}^n \frac{1}{\mu_i^\delta} \frac{1}{\mu_j^\delta} (L_{ii} P_{jj}^N - L_{ij} P_{ij}^N + P_{ii}^N L_{jj} - P_{ij}^N L_{ij}) \\
&= \frac{1}{\mu_n^\delta} \sum_{i=1}^n \frac{1}{\mu_i^\delta} \lambda_i = \sum_{i=1}^{n-1} \frac{1}{\mu_i^\delta} \lambda_i
\end{aligned} \tag{7.9}$$

Note that the eigenvalues $\mu_i^\delta \rightarrow 1$ since $g_\delta \rightarrow g$ uniformly, hence $\text{tr}_g(\mathbf{L}) = \sum_{i=1}^n \lambda_i > 0$ implies

$$\sum_{i=1}^{n-1} \frac{1}{\mu_i^\delta} \lambda_i \geq (1 - \varepsilon(\delta)) \sum_{i=1}^{n-1} \lambda_i \geq 0$$

for small enough δ . We then proceed as in the previous section and estimate the f'_δ -term in (7.8) from below by $-\varepsilon(\delta)$, which completes the proof for the case $\text{tr}_g L > 0$ on Γ .

Let us now study the case where $\text{tr}_g L \geq 0$ on Γ . In this case we may slightly modify either one of the initial metrics g_0 or g_1 near the boundary, such that $\text{tr}_g L$ becomes strictly positive, and then repeat the argument above. More precisely, consider g_0 near Γ . Recall that in local coordinates (x^1, \dots, x^n) we chose in Section 2, g_0 has the form

$$g_0 = \begin{pmatrix} \hat{g}_0 & 0 \\ 0 & 1 \end{pmatrix}$$

where \hat{g} is the restriction of g to the equidistant hypersurfaces $\Gamma(d)$, $d = \text{dist}_g(\Gamma, \cdot) = x^n$. Let $d_0 > 0$ small enough so that $\Gamma(d)$ is smooth for $d \leq d_0$. We find a smooth function $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$\varphi(0) = 1, \varphi'(0) < 0, \varphi|_{[d_0, \infty)} \equiv 1 \text{ and } |\varphi'|, |\varphi''| \leq \varepsilon$$

with ε small, and put

$$\tilde{g}_0 = \begin{pmatrix} \varphi(x^n) \hat{g}_0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that in view of $\varphi(0) = 1$ we have $\tilde{g}_0|_\Gamma = g_0|_\Gamma = g_1|_\Gamma$, so that the isometry of the boundaries is preserved. As in Lemma 2.8, in a point $p \in \Gamma$ (i.e. $x^n(p) = 0$) we compute

$$\tilde{L}_{ij}^0 = -\frac{1}{2} \partial_n \tilde{g}_{ij}^0 = -\frac{1}{2} \varphi'(0) g_{ij}^0 - \frac{1}{2} \varphi(0) \partial_n g_{ij}^0 = -\frac{1}{2} \varphi'(0) g_{ij}^0 + L_{ij}^0$$

and thus

$$\mathrm{tr}_{\tilde{g}_0}(\tilde{L}_0) = g_0^{ij}(-\frac{1}{2}\varphi'(0)g_{ij}^0 + L_{ij}^0) = -\frac{n}{2}\varphi'(0) + \mathrm{tr}_{g_0}L_0 > \mathrm{tr}_{g_0}L_0$$

which gives us $\mathrm{tr}_{\tilde{g}_0}\tilde{L}_0 + \mathrm{tr}_{g_1}L_1 > 0$, since by assumption $\mathrm{tr}_{g_0}L_0 + \mathrm{tr}_{g_1}L_1 = \mathrm{tr}_gL \geq 0$ on Γ . Moreover, by construction, the new metric \tilde{g}_0 is C^2 -close to g_0 , thus their scalar curvatures differ only by an error term ε coming from the first two derivatives of φ , which we may choose arbitrary small. We then may replace g_0 by \tilde{g}_0 and proceed like in the $\mathrm{tr}_gL > 0$ case. \square

Remark: In [3] P. Miao generalized the positive mass theorem [6] (which says that an asymptotically flat manifold with nonnegative scalar curvature has nonnegative ADM mass), to metrics which fail to be C^1 across a hypersurface Σ . One of the essential steps of his proof was to smoothen the metric across Σ in such a way that the scalar curvature stays bounded from below by a constant (cf. [3], Proposition 3.1). Theorem 7.2 provides a slightly better approximation, since in our case the smooth metrics have scalar curvature $\geq -\varepsilon$.

7.3 Manifolds with bi-curvature $\geq \kappa$

The bi-curvature $\mathrm{bi}(g)$ of a C^2 -smooth Riemannian metric g is defined as the sum of the two smallest eigenvalues of $\mathcal{R}(g)$. Note that $\mathrm{bi}(g) \geq \kappa$ holds on M iff

$$\mathcal{R}(g)(\alpha, \alpha) + \mathcal{R}(g)(\beta, \beta) \geq \kappa$$

for all $\alpha, \beta \in \Lambda^2(TM)$ which are orthonormal with respect to g .

Theorem 7.3. *Let (M_0, g_0) , (M_1, g_1) , (M, g) and $L = L_0 + L_1$ be as in Theorem 1.2. Suppose that $\mathrm{bi}(g_0)$ and $\mathrm{bi}(g_1)$ are at least κ . If L is positive semidefinite, then $\mathrm{bi}(g) \geq \kappa$ (in a similar sense as in Definition 1.1).*

Proof. We proceed as in the previous section and show

(a) The modified metric g_δ on M_0 satisfies $\mathrm{bi}(g_\delta) \geq \kappa - \varepsilon(\delta)$ where $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$

(b) By mollifying $g_{(\delta)}$ we construct a family of smooth metrics which approximate g in the C^0 -sense and have bi-curvature at least $\kappa - \varepsilon(\delta)$.

As mentioned above, (a) holds iff

$$\mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \geq \kappa - \varepsilon(\delta) \quad (7.10)$$

for all $\alpha_\delta, \beta_\delta$ satisfying $\|\alpha_\delta\|_\delta, \|\beta_\delta\|_\delta = 1$ and $\langle \alpha_\delta, \beta_\delta \rangle_\delta = 0$ (where $\langle \cdot, \cdot \rangle_\delta = \mathcal{I}(g_\delta)$). In what follows we will call such 2-forms g_δ -orthonormal. Theorem 3.3 implies

$$\begin{aligned} & \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\ &= \mathcal{R}(\alpha_\delta, \alpha_\delta) + \mathcal{R}(\beta_\delta, \beta_\delta) - f_\delta^2(\mathcal{A}(\alpha_\delta, \alpha_\delta) + \mathcal{A}(\beta_\delta, \beta_\delta)) + f_\delta(\mathcal{B}(\alpha_\delta, \alpha_\delta) + \mathcal{B}(\beta_\delta, \beta_\delta)) \\ &- 2f'_\delta(\mathcal{L}(\alpha_\delta, \alpha_\delta) + \mathcal{L}(\beta_\delta, \beta_\delta)) + 2f_\delta^2(\mathcal{L}^2(\alpha_\delta, \alpha_\delta) + \mathcal{L}^2(\beta_\delta, \beta_\delta)) + 2Cf_\delta(\hat{\mathcal{I}}(\alpha_\delta, \alpha_\delta) + \hat{\mathcal{I}}(\beta_\delta, \beta_\delta)) \\ &+ (\mathcal{E}(\delta)(\alpha_\delta, \alpha_\delta) + \mathcal{E}(\delta)(\beta_\delta, \beta_\delta)) \end{aligned}$$

where $\mathcal{E}(\delta)$ is an operator whose eigenvalues tend to zero uniformly on M_0 . Since $g_\delta \rightarrow g_0$ uniformly on M_0 , for small enough δ any g_δ -orthonormal forms α_δ and β_δ are uniformly bounded with respect to g_0 by some fixed constant. Thus, we can estimate the $\mathcal{E}(\delta)$ terms from below by $-\varepsilon(\delta)$. \mathcal{L} is positive semidefinite and bounded near Γ , and f'_δ does not exceed δ . Therefore, $-2f'_\delta(\mathcal{L}(\alpha_\delta, \alpha_\delta) + \mathcal{L}(\beta_\delta, \beta_\delta)) \geq -\varepsilon(\delta)$. Finally, the \mathcal{L}^2 terms are nonnegative and we arrive at

$$\begin{aligned} & \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\ &\geq \mathcal{R}(\alpha_\delta, \alpha_\delta) + \mathcal{R}(\beta_\delta, \beta_\delta) - f_\delta^2(\mathcal{A}(\alpha_\delta, \alpha_\delta) + \mathcal{A}(\beta_\delta, \beta_\delta)) + f_\delta(\mathcal{B}(\alpha_\delta, \alpha_\delta) + \mathcal{B}(\beta_\delta, \beta_\delta)) \\ &+ Cf_\delta(\hat{\mathcal{I}}(\alpha_\delta, \alpha_\delta) + \hat{\mathcal{I}}(\beta_\delta, \beta_\delta)) - \varepsilon(\delta) \end{aligned} \quad (7.11)$$

By applying the Gram-Schmidt process to α_δ and β_δ and putting

$$\tilde{\alpha}_\delta := \frac{\alpha_\delta}{\|\alpha_\delta\|_0}$$

and

$$\tilde{\beta}_\delta := \frac{\beta_\delta - \langle \tilde{\alpha}_\delta, \beta_\delta \rangle_0 \tilde{\alpha}_\delta}{\|\beta_\delta - \langle \tilde{\alpha}_\delta, \beta_\delta \rangle_0 \tilde{\alpha}_\delta\|_0}$$

we obtain g_0 -orthonormal 2-forms $\tilde{\alpha}_\delta, \tilde{\beta}_\delta$ satisfying

$$\|\tilde{\alpha}_\delta - \alpha_\delta\|_0, \|\tilde{\beta}_\delta - \beta_\delta\|_0 \leq \varepsilon(\delta)$$

independent of the initial $\alpha_\delta, \beta_\delta$. Since f_δ and all of the operators on the right hand side of (7.11) are uniformly bounded near Γ we may replace $\alpha_\delta, \beta_\delta$ by $\tilde{\alpha}_\delta, \tilde{\beta}_\delta$ and the inequality will still hold up to an $-\varepsilon(\delta)$:

$$\begin{aligned} & \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\ \geq & \mathcal{R}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{R}(\tilde{\beta}_\delta, \tilde{\beta}_\delta) - f_\delta^2(\mathcal{A}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{A}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)) + f_\delta(\mathcal{B}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \mathcal{B}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)) \\ & + 2Cf_\delta(\hat{\mathcal{I}}(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + \hat{\mathcal{I}}(\tilde{\beta}_\delta, \tilde{\beta}_\delta)) - \varepsilon(\delta) \end{aligned}$$

From construction $\tilde{\alpha}_\delta$ and $\tilde{\beta}_\delta$ are g_0 -orthonormal on M_0 and g_1 -orthonormal on Γ (recall that $g_0 = g_1$ on Γ). By adopting the argument from Lemma 5.1 we arrive at

$$\begin{aligned} & \mathcal{R}_\delta(\alpha_\delta, \alpha_\delta) + \mathcal{R}_\delta(\beta_\delta, \beta_\delta) \\ \geq & \kappa + 2f_\delta\left[(-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}})(\tilde{\alpha}_\delta, \tilde{\alpha}_\delta) + (-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}})(\tilde{\beta}_\delta, \tilde{\beta}_\delta)\right] \\ & - \varepsilon(\delta) \end{aligned}$$

Since $-\mathcal{L}^2 + \frac{1}{2}\nabla_N^2\mathcal{G}_1 + C\hat{\mathcal{I}}$ is positive semidefinite for large enough fixed C and uniformly bounded near Γ (cf. proof of Lemma 5.2), (a) follows.

(b) Mollifying $g_{(\delta)}$:

Let us fix a $\delta > 0$ and define the mollified metric $g_{(\delta)}^h$ in the same way as in Section 6. Our goal is to show

$$\mathcal{R}(g_{(\delta)}^h)(\alpha, \alpha) + \mathcal{R}(g_{(\delta)}^h)(\beta, \beta) \geq \kappa - \varepsilon(\delta) \quad (7.12)$$

for all $g_{(\delta)}^h$ -orthonormal α, β . The computations in Section 6 were carried out for 2-forms with constant coefficients, which we no longer can assume for orthonormal 2-forms.

Using (6.3) we obtain

$$\mathcal{R}(g_{(\delta)}^h)(\alpha, \alpha) + \mathcal{R}(g_{(\delta)}^h)(\beta, \beta) \geq (\mathcal{R}(g_{(\delta)}))^h(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(\beta, \beta) - \tilde{\varepsilon}(\delta, h)(\|\alpha\|_{(\delta)}^2 + \|\beta\|_{(\delta)}^2)$$

where $\tilde{\varepsilon}(\delta, h) \rightarrow 0$ as $h \rightarrow 0$ for every fixed δ . Since α and β have unit length with respect to $g_{(\delta)}^h$ and $g_{(\delta)}^h \xrightarrow{h \rightarrow 0} g_{(\delta)}$, we can estimate the last term on the right hand side from below by $-\varepsilon(\delta)$ for small enough h . Thus, (7.12) follows if we show

$$(\mathcal{R}(g_{(\delta)}))^h(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(\beta, \beta) \geq \kappa - \tilde{\varepsilon}(\delta) \quad (7.13)$$

for small enough h and all $g_{(\delta)}^h$ -orthonormal α, β .

Let us now fix a point $x \in M$ and some $g_{(\delta)}^h(x)$ -orthonormal $\alpha, \beta \in \Lambda^2(T_x M)$. Recall that in Section 6 we mollified $g_{(\delta)}$ and $R(g_{(\delta)})$ only on a small neighborhood of Γ which was covered by finitely many coordinate charts U_1, \dots, U_N . Off this neighborhood g coincides with $g_{(\delta)}$ and we have

$$(\mathcal{R}(g_{(\delta)}))^h(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(\beta, \beta) = \mathcal{R}(g)(\alpha, \alpha) + \mathcal{R}(g)(\beta, \beta) \geq \kappa$$

by assumption. Thus w.l.o.g. we can assume that $x \notin \bigcup_{s>N} U_s$. For such x we have

$$\begin{aligned} & (\mathcal{R}(g_{(\delta)}))^h(x)(\alpha, \alpha) + (\mathcal{R}(g_{(\delta)}))^h(x)(\beta, \beta) \\ &= \sum_{s=1}^N \eta_s(x) \int_{|z| \leq 1} \rho(z) (R(g_{(\delta)}))^s_{ijkl}(x - hz) (\alpha_s^{ij} \alpha_s^{kl} + \beta_s^{ij} \beta_s^{kl}) dz \end{aligned} \quad (7.14)$$

where the coefficients refer to the charts (U'_s, φ_s) . Next we extend α, β to U'_s in such a way that the extensions are $g_{(\delta)}^h$ -orthonormal: We define 2-forms α_s, β_s on U'_s , $s = 1, \dots, N$ by putting $\alpha_s^{ij}(y) := \alpha_s^{ij}$ and $\beta_s^{ij}(y) := \beta_s^{ij}$ (here we only have to consider the neighborhoods with $x \in U_s$). Using the Gram-Schmidt process we obtain $g_{(\delta)}^h$ -orthonormal 2-forms

$$\tilde{\alpha}_s = \frac{\alpha_s}{\|\alpha_s\|_{g_{(\delta)}^h}}$$

and

$$\tilde{\beta}_s = \frac{\beta_s - \langle \tilde{\alpha}_s, \beta_s \rangle_{g_{(\delta)}^h} \tilde{\alpha}_s}{\|\beta_s - \langle \tilde{\alpha}_s, \beta_s \rangle_{g_{(\delta)}^h} \tilde{\alpha}_s\|_{g_{(\delta)}^h}}$$

(Note that these extensions might differ on $U_s \setminus \{x\}$.) By the mean value theorem the right hand side of (7.14) equals to

$$\begin{aligned} & \sum_{s=1}^N \eta_s(x) \int_{|z| \leq 1} \rho(z) (R(g_{(\delta)}))^s_{ijkl}(x - hz) (\tilde{\alpha}_s^{ij}(x - hz) \tilde{\alpha}_s^{kl}(x - hz) + \tilde{\beta}_s^{ij}(x - hz) \tilde{\beta}_s^{kl}(x - hz)) dz \\ &+ h \sum_{s=1}^N \eta_s(x) \int_{|z| \leq 1} \rho(z) (R(g_{(\delta)}))^s_{ijkl}(x - hz) D(\tilde{\alpha}_s^{ij} \tilde{\alpha}_s^{kl} + \tilde{\beta}_s^{ij} \tilde{\beta}_s^{kl})(\xi_{x,hz}^s) z dz \end{aligned} \quad (7.15)$$

where $\xi_{x,hz}^s = (1-t)x + thz$ for some $t \in [0, 1]$. Now we apply the Gram-Schmidt process with respect to $g_{(\delta)}$ to the 2-forms $\tilde{\alpha}_s$ and $\tilde{\beta}_s$, and construct $g_{(\delta)}$ -orthonormal $\tilde{\tilde{\alpha}}_s, \tilde{\tilde{\beta}}_s$. The first sum in (7.15) is estimated from below by

$$\begin{aligned} & \sum_{s=1}^N \eta_s(x) \int_{|z| \leq 1} \rho(z) (R(g_{(\delta)}))^s_{ijkl}(x - hz) (\tilde{\tilde{\alpha}}_s^{ij}(x - hz) \tilde{\tilde{\alpha}}_s^{kl}(x - hz) + \tilde{\tilde{\beta}}_s^{ij}(x - hz) \tilde{\tilde{\beta}}_s^{kl}(x - hz)) \\ &- \varepsilon(\delta, h) \end{aligned} \quad (7.16)$$

where

$$\varepsilon(\delta, h) \leq c(n) \|(R(g_{(\delta)}))^s_{ijkl}\|_{L^\infty(U'_s)} \|(g_{(\delta)} - g_{(\delta)}^h)_{ij}^s\|_{C^0(U'_s)} \xrightarrow{h \rightarrow 0} 0$$

for every fixed δ . Moreover, in view of (a) the integrand in (7.16) is bounded from below by $\kappa - \varepsilon(\delta)$. Finally, the second integrand in (7.15) is bounded by

$$c(n) \|(R(g_{(\delta)}))^s_{ijkl}\|_{L^\infty(U'_s)} \|(g_{(\delta)} - g_{(\delta)}^h)_{ij}^s\|_{C^1(U'_s)}$$

and thus the second expression in (7.15) tends to zero uniformly as $h \rightarrow 0$. For small enough h (7.13) follows with $\tilde{\varepsilon}(\delta) = 2\varepsilon(\delta)$ and we are done. \square

7.4 Manifolds with isotropic curvature $\geq \kappa$

Given a smooth Riemannian manifold (M, g) we consider the complexification of its tangent bundle $\mathbb{C} \otimes_{\mathbb{R}} TM$ and the complex-linear extensions of the inner product g and the Riemannian curvature tensor R . A complex isotropic two-plane is spanned by two vectors $Z = X + iY$ and $W = U + iV$ where $X, Y, U, V \in TM$ are orthonormal with respect to g . The isotropic curvature of such a two-plane P is defined as

$$K(P) = R(Z, W, \bar{Z}, \bar{W})$$

Using the Bianchi identity

$$R(X, Y, U, V) + R(X, V, Y, U) + R(X, U, V, Y) = 0$$

one easily verifies

$$K(P) = R(X, U, X, U) + R(X, V, X, V) + R(Y, U, Y, U) + R(Y, V, Y, V) - 2R(X, Y, U, V)$$

Given an isotropic two-plane P spanned by $X + iY$ and $U + iV$ one computes using the Bianchi identity once more

$$K(P) = \mathcal{R}(\alpha, \alpha) + \mathcal{R}(\beta, \beta) \quad (7.17)$$

where $\alpha = X \wedge U + V \wedge Y$ and $\beta = X \wedge V + Y \wedge U$. We say that a Riemannian manifold has isotropic curvature $\geq \kappa$ if $K(P) \geq \kappa$ holds for all isotropic two-planes of M . M has 1-isotropic (2-isotropic) curvature $\geq \kappa$ if $M \times \mathbb{R}$ ($M \times \mathbb{R}^2$) has isotropic curvature $\geq \kappa$.

Theorem 7.4. *Let (M_0, g_0) , (M_1, g_1) , (M, g) and $L = L_0 + L_1$ be as in Theorem 1.2. Suppose that the isotropic (1-isotropic, 2-isotropic) curvatures of g_0 and g_1 are at least κ . Then the isotropic (1-isotropic, 2-isotropic) curvatures of g is at least κ (in a similar sense as in Definition 1.1) if L is positive semidefinite.*

Proof. In view of (7.17), the proof for the isotropic case is similar as in the previous section. For the 1-isotropic case let us examine the manifold resulting from gluing $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ along their boundaries. The boundary of M_i , $i = 1, 2$ is given by $\Gamma_i \times \mathbb{R}$. If $\phi : \Gamma_1 \rightarrow \Gamma_2$ is some isometry of Γ_1, Γ_2 with respect to g_1, g_2 , then

$$\begin{aligned} \tilde{\phi} : \Gamma_1 \times \mathbb{R} &\rightarrow \Gamma_2 \times \mathbb{R} \\ (x, s) &\mapsto (\phi(x), s) \end{aligned}$$

is an isometry of $\Gamma_1 \times \mathbb{R}$ and $\Gamma_2 \times \mathbb{R}$ with respect to $g_0 \oplus dr$, $g_1 \oplus dr$, where dr denotes the standard metric on \mathbb{R} . One easily verifies that

$$(M_1 \times \mathbb{R}) \cup_{\tilde{\phi}} (M_2 \times \mathbb{R}) = (M_1 \cup_{\phi} M_2) \times \mathbb{R}$$

and

$$(g \oplus dr)|_{M_i \times \mathbb{R}} = g|_{M_i} \oplus dr$$

The inward normal on $\Gamma_i \times \mathbb{R}$ with respect to $g_i \oplus dr$ is given by $(N, 0)$, where N is the inward normal on Γ_i with respect to g_i . The second fundamental forms of $\Gamma_i \times \mathbb{R}$ are $L_i \oplus 0$, and therefore their sum is positive semidefinite. We repeat the constructions from Section 2 and define the modified metric $(g_0 \oplus dr)_\delta = g_\delta \oplus dr$ on $M_0 \times \mathbb{R}$. Though $\Gamma \times \mathbb{R}$ fails to be compact, we may nevertheless proceed as in the isotropic case, since any operator \mathcal{T} which occurs for $M_i \times \mathbb{R}$ during the proof satisfies $\mathcal{T}(x, s) = \mathcal{T}(x, 0)$, and therefore is bounded near $\Gamma \times \mathbb{R}$ due to the compactness of Γ . The desired smooth metric on $M \times \mathbb{R}$, which approximates $g \oplus dr$ and has isotropic curvature $\geq \kappa - \varepsilon(\delta)$ is then given by $g_{(\delta)} \oplus dr$. The proof for the 2-isotropic case is similar. □

7.5 Manifolds with flag curvature $\geq \kappa$

The flag curvature of an orthonormal three-frame $\{e_1, e_2, e_3\}$ is defined as

$$\text{flag}(e_1, e_2, e_3) = \mathcal{R}(e_1 \wedge e_3, e_1 \wedge e_3) + \mathcal{R}(e_2 \wedge e_3, e_2 \wedge e_3)$$

Theorem 7.5. *Let (M_0, g_0) , (M_1, g_1) , (M, g) and $L = L_0 + L_1$ be as in Theorem 1.2. Suppose that the flag curvatures of g_0 and g_1 are at least κ . Then the flag curvatures of g is at least κ (in a similar sense as in Definition 1.1) if L is positive semidefinite.*

The proof is similar as in the bi-curvature case.

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